T_N, Toda and Topological strings

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Strings 2019

Brussels, 9 July

[1310.3841 Bao,Mitev,EP,Taki,Yagi]
[1409.6313 Mitev,EP]
[1412.3395 Isachenkov, Mitev,EP]

[1712.10225 Coman,EP,Teschner]
[1906.06351 Coman,EP,Teschner]

T_N theories

Class S *of 4D $\mathcal{N}=2$ SCFT $T_{g,n}$ by a compactification of the A_N 6D (2,0) SCFT on Riemann surface of genus g and with n punctures.* [Gaiotto 2009]

*Riemann surface decomposition in **pants** and **tubes**.*

*Building blocks of Gauge theories: **matter** and **color factors.***

T_N theories are the most general (trifundamental) "matter".

- *$SU(N)^3$ global symmetry*
- *Isolated fixed points*
- *No $\mathcal{N}=2$ Lagrangian description (N>2)*

One open problem (2 incarnations)

The partition function of T_N theories on S^4

↕ *AGT* [Alday,Gaiotto,Tachikawa 2009]

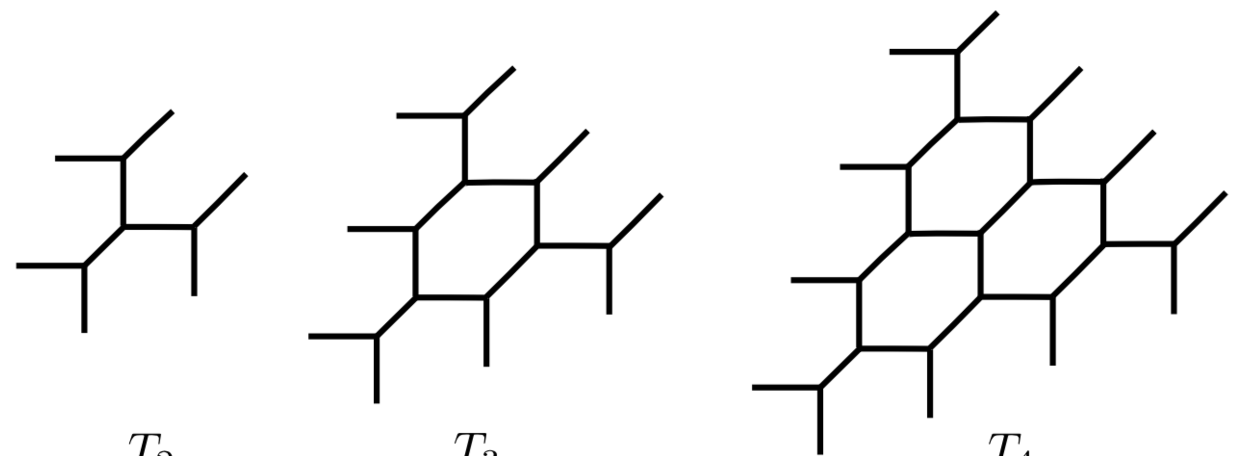
3pt functions of Toda with ***3 generic primaries***

*Use topological strings, pay **a price**:*

5D partition function on $S^4 \times S^1$ ⟷ *q-deformation of Toda CFT*

*The 4D or $q \to 1$ **limit is hard!*** $q = e^{-R}$

Outline



1 $\mathcal{Z}_{T_N}^{S^4 \times S^1}$ from web diagrams using the ***topological vertex formalism.***

[1310.3841 Bao,Mitev,EP,Taki,Yagi]
[1310.3854 Hayashi,Kim,Nishinaka]

2 Proposal for 3pt functions of q-Toda: ***3 generic primaries.***

- ☑ N=2 Liouville case
- ☑ symmetries, zeros

[1409.6313 Mitev,EP]
[1412.3395 Isachenkov,Mitev,EP]

3 *Important check*: reproduce ***free field integral representation!*** *

And good for $q \to 1$*!*

- ☑ poles

[1906.06351 Coman,EP,Teschner]

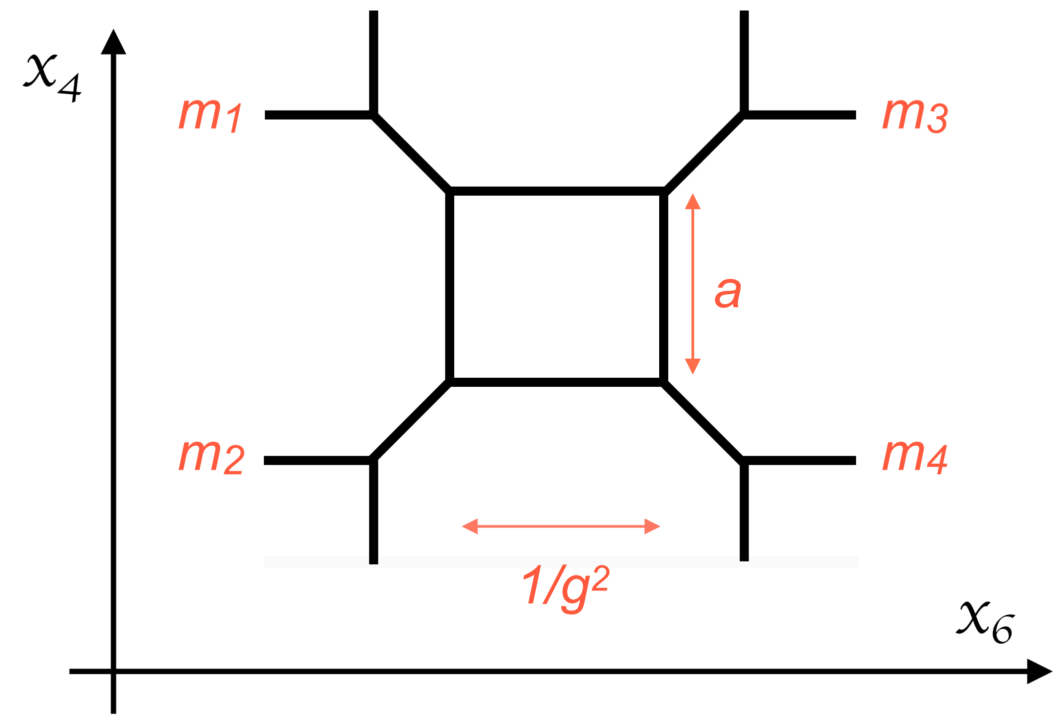
* Up to contours!

5-brane Webs

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
NS5-branes	—	—	—	—	—	—	.	.	.	.
D5-branes	—	—	—	—	.	—	—	.	.	.

The low energy dynamics of 5D theories is encoded in web diagrams:

ex.
SU(2)
with N_f=4



5D theories on S^1

$x_5 \sim x_5 + \mathcal{R}$

- *# of external branes - 3 = masses and couplings (m's and g's) = 8 - 3 = 4 + 1*
- *# of faces = Coulomb branch (a's) = 1*

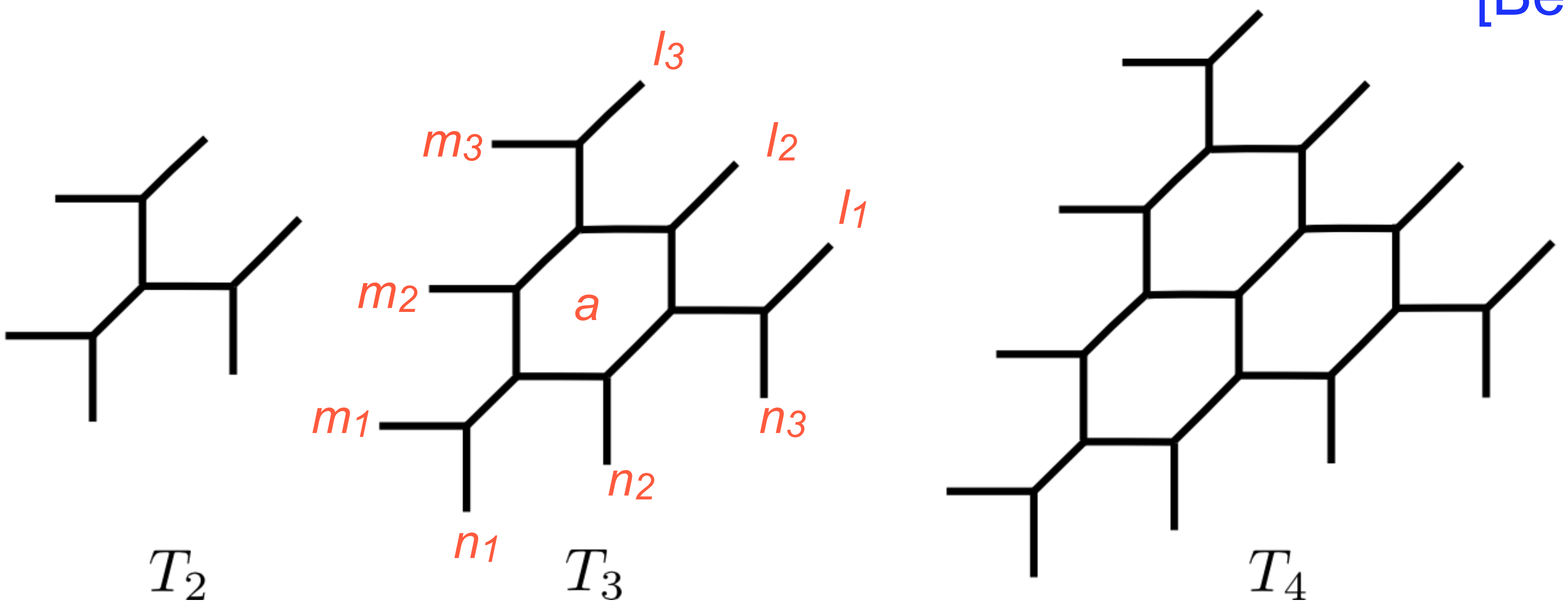
$$\mathcal{Z}_{\text{Nek}}^{\mathbb{R}^4\times S^1} \propto \mathcal{Z}_{\text{top}}$$

5D Nekrasov from the topological vertex formalism

5-brane Webs

The low energy dynamics of 5D T_N theories is encoded in the web diagrams:

[Benini,Benvenuti,Tachikawa]



- *# of external branes - 3 = masses = 3(N-1) (No coupling)*
- *# of faces = Coulomb branch (a's) = (N-1)(N-2)/2*

The T_N topological string partition function using the topological vertex formalism

Extra degrees of freedom

[Bao,Mitev,EP,Taki,Yagi]
[Hayashi,Kim,Nishinaka]

When the web diagram has parallel external legs, Z_{top} includes extra d.o.f.

$$\mathcal{Z}_{\text{Nek}}^{\mathbb{R}^4 \times S^1} = \frac{\mathcal{Z}_{\text{top}}}{\mathcal{Z}_{\text{extra}}}$$

*Only after removing them we get the correct 5D partition function (with **symmetry enhancement**)*

ex. T_3 has E_6 [Argyres,Seiberg]

SU(2) with N_f flavours has E_{N_f+1}
[Seiberg 1996]

For the T_N:

$$\mathcal{Z}_{\text{extra}} = \prod_{i<j=1}^{N} \mathcal{M}(e^{-R(m_i - m_j)}) \mathcal{M}(e^{-R(n_i - n_j - \epsilon_1 - \epsilon_2)}) \mathcal{M}(e^{-R(\ell_i - \ell_j)})$$

$$\mathcal{M}(e^{-Ru}) = \prod_{i,j=1}^{\infty} \left(1 - e^{-R[u + j\epsilon_1 - (i-1)\epsilon_2]}\right)^{-1}$$

T_N partition functions

[Bao,Mitev,EP,Taki,Yagi]
[Hayashi,Kim,Nishinaka]

$$\mathcal{Z}_{T_N}^{S^4\times S^1} = \frac{1}{|\mathcal{Z}_{\text{extra}}|^2} \oint \prod_{\text{faces}} [da]\, |\mathcal{Z}_{\text{top}}|^2$$

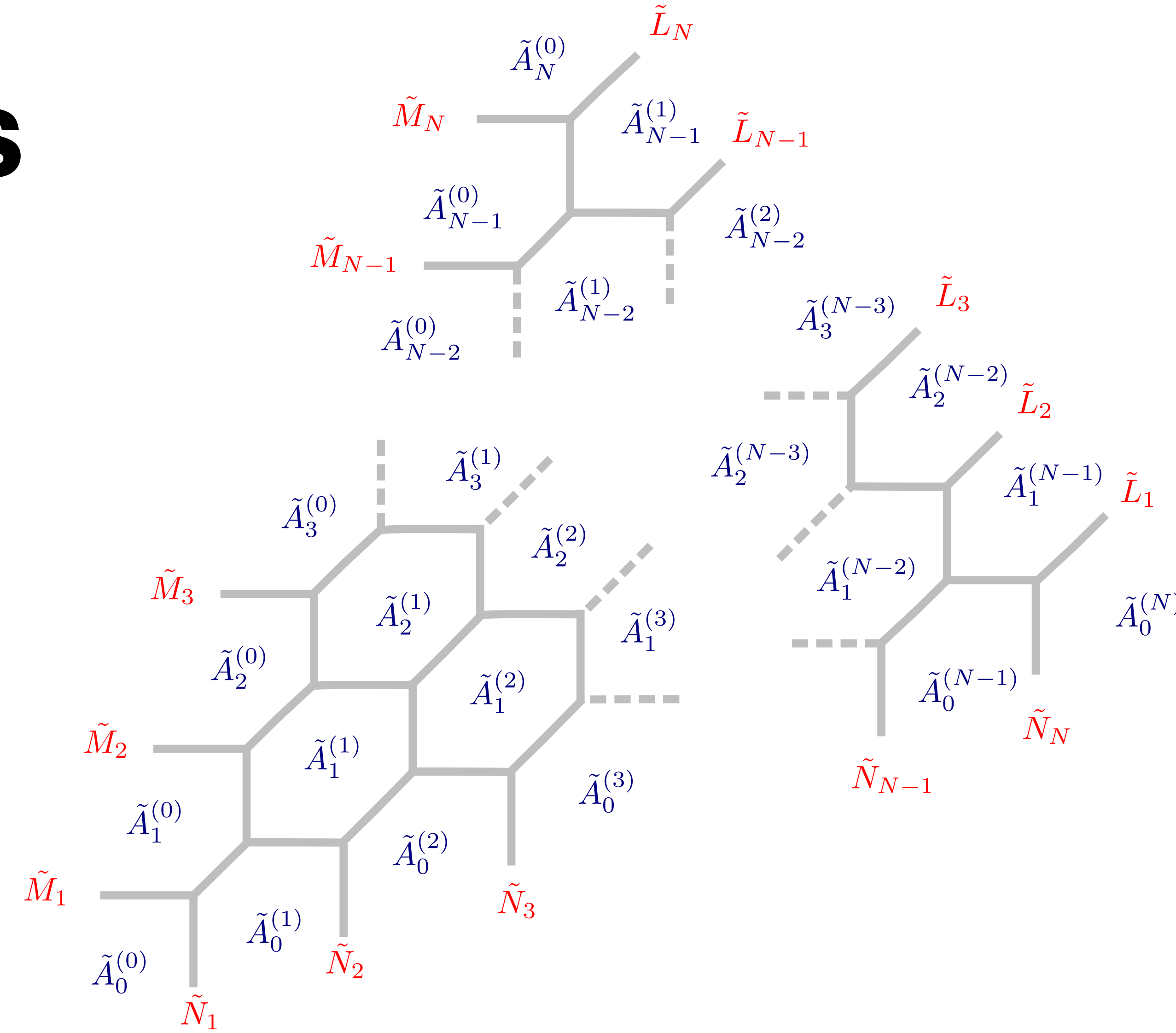
[Iqbal,Vafa 2012]

- (N-1)(N-2)/2 integrals
  = # of faces

$$\mathcal{Z}_{\text{top}} = \mathcal{Z}_{\text{prod}}\mathcal{Z}_{\text{sum}}$$

$\mathcal{Z}_{\text{prod}}$ *very similar to the usual perturbative part of Lagrangian theories.*

T_N partition functions



$$\mathcal{Z}_{\text{top}} = \mathcal{Z}_{\text{prod}}\mathcal{Z}_{\text{sum}}$$

$$\tilde{A} = e^{-Ra} \qquad \mathfrak{q} = e^{-R\epsilon_1}\,,\ \mathfrak{t} = e^{R\epsilon_2}$$

$$\mathcal{M}(U) = \prod_{i,j=1}^{\infty}\left(1 - U\mathfrak{t}^{i-1}\mathfrak{q}^{j}\right)^{-1}$$

$$\mathcal{Z}_N^{\text{prod}} = \prod_{r=1}^{N-1}\prod_{i\le j=1}^{N-r}\frac{\mathcal{M}\left(\frac{\tilde{A}_i^{(r-1)}\tilde{A}_j^{(r-1)}}{\tilde{A}_{i-1}^{(r-1)}\tilde{A}_{j+1}^{(r-1)}}\right)}{\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{A}_i^{(r-1)}\tilde{A}_{j-1}^{(r)}}{\tilde{A}_{i-1}^{(r-1)}\tilde{A}_j^{(r)}}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{A}_i^{(r)}\tilde{A}_j^{(r-1)}}{\tilde{A}_{i-1}^{(r)}\tilde{A}_{j+1}^{(r-1)}}\right)}\prod_{i\le j=1}^{N-r-1}\mathcal{M}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{A}_i^{(r)}\tilde{A}_j^{(r)}}{\tilde{A}_{i-1}^{(r)}\tilde{A}_{j+1}^{(r)}}\right)$$

[Bao,Mitev,EP,Taki,Yagi]

[Hayashi,Kim,Nishinaka]

$$\mathsf{N}_{\lambda\mu}^{R}(a) = \prod_{(i,j)\in\lambda}2\sinh\frac{R}{2}\left[a+\epsilon_1(\lambda_i-j+1)+\epsilon_2(i-\mu_j^t)\right]\prod_{(i,j)\in\mu}2\sinh\frac{R}{2}\left[a+\epsilon_1(j-\mu_i)+\epsilon_2(\lambda_j^t-i+1)\right]$$

$$\mathcal{Z}_N^{\text{sum}} = \sum_{\boldsymbol{\nu}}\prod_{r=1}^{N}\prod_{i=1}^{N-r}\left(\frac{\tilde{N}_r\tilde{L}_{N-r}}{\tilde{N}_{r+1}\tilde{L}_{N-r+1}}\right)^{\frac{|\nu_i^{(r)}|}{2}}\prod_{r=1}^{N}\prod_{i\le j=1}^{N-r}\left[\frac{\mathsf{N}^R_{\nu_i^{(r-1)}\nu_j^{(r)}}\left(a_i^{(r-1)}+a_{j-1}^{(r)}-a_{i-1}^{(r-1)}-a_j^{(r)}-\epsilon_+/2\right)}{\mathsf{N}^R_{\nu_i^{(r-1)}\nu_{j+1}^{(r-1)}}\left(a_i^{(r-1)}+a_j^{(r-1)}-a_{i-1}^{(r-1)}-a_{j+1}^{(r-1)}\right)}\frac{\mathsf{N}^R_{\nu_i^{(r)}\nu_{j+1}^{(r-1)}}\left(a_i^{(r)}+a_j^{(r-1)}-a_{i-1}^{(r)}-a_{j+1}^{(r-1)}-\epsilon_+/2\right)}{\mathsf{N}^R_{\nu_i^{(r)}\nu_j^{(r)}}\left(a_i^{(r)}+a_{j-1}^{(r)}-a_{i-1}^{(r)}-a_j^{(r)}-\epsilon_+\right)}\right]$$

T_N partition functions

[Bao,Mitev,EP,Taki,Yagi]

[Hayashi,Kim,Nishinaka]

$$\mathcal{Z}_{T_N}^{S^4 \times S^1} = \frac{1}{|\mathcal{Z}_{\text{extra}}|^2} \oint \prod_{\text{faces}} [da] \, |\mathcal{Z}_{\text{top}}|^2$$

- (N-1)(N-2)/2 integrals = # of faces

$$\mathcal{Z}_{\text{top}} = \mathcal{Z}_{\text{prod}} \mathcal{Z}_{\text{sum}}$$

- N(N-1)/2 sums still left to perform

$\mathcal{Z}_{\text{prod}}$ *very similar to the usual perturbative part of Lagrangian theories.*

$\mathcal{Z}_{\text{sum}}$ *very similar to the usual instanton part of Lagrangian theories.*

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$\mathcal{Z}_{\text{prod}}$ *very similar to the usual* ***perturbative part*** *of Lagrangian theories.*

$\mathcal{Z}_{\text{sum}}$ *very similar to the usual* ***instanton part*** *of Lagrangian theories.*

with one ***crucial difference****:*

$$\sum_\mu (q_{UV}^{5D})^{|\mu|} \frac{\mathsf{N}^R_{\mu\nu_1}(a_1)\cdots \mathsf{N}^R_{\mu\nu_L}(a_L)}{\mathsf{N}^R_{\mu\lambda_1}(b_1)\cdots \mathsf{N}^R_{\mu\lambda_L}(b_L)} \xrightarrow{R\to 0} \sum_\mu (q_{UV}^{4D})^{|\mu|} \frac{\mathsf{N}_{\mu\nu_1}(a_1)\cdots \mathsf{N}_{\mu\nu_L}(a_L)}{\mathsf{N}_{\mu\lambda_1}(b_1)\cdots \mathsf{N}_{\mu\lambda_L}(b_L)}$$

For Lagrangian theories the 4D limit OK!

$$\sum_\mu (e^{-Rx})^{|\mu|} \frac{\mathsf{N}^R_{\mu\nu_1}(a_1)\cdots \mathsf{N}^R_{\mu\nu_L}(a_L)}{\mathsf{N}^R_{\mu\lambda_1}(b_1)\cdots \mathsf{N}^R_{\mu\lambda_L}(b_L)} \xrightarrow{R\to 0} \sum_\mu (1)^{|\mu|} \frac{\mathsf{N}_{\mu\nu_1}(a_1)\cdots \mathsf{N}_{\mu\nu_L}(a_L)}{\mathsf{N}_{\mu\lambda_1}(b_1)\cdots \mathsf{N}_{\mu\lambda_L}(b_L)}$$

For T_N problematic!

In the 4D limit the sum may not even converge!

2D CFT Review

$\Upsilon(x) \sim \mathrm{Reg}\left[\prod_{n_1,n_2\geq 0}(x + b\,n_1 + b^{-1}n_2)\,(-x + b\,(n_1+1) + b^{-1}(n_2+1))\right]$

satisfies the shift relations $\begin{cases}\Upsilon(x+b) = \gamma(bx)\,b^{1-2bx}\,\Upsilon(x)\\ \Upsilon(x+b^{-1}) = \gamma(b^{-1}x)\,b^{2b^{-1}x-1}\,\Upsilon(x)\end{cases}$

For N=2 (Liouville) DOZZ: [Dorn,Otto] [Zamolodchikov^2]

$$C(\alpha_1,\alpha_2,\alpha_3) = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{Q-\sum_{i=1}^3 \alpha_i}{b}} \frac{\Upsilon'(0)\prod_{i=1}^3 \Upsilon(2\alpha_i)}{\Upsilon(\sum_{i=1}^3 \alpha_i - Q)\prod_{j=1}^3 \Upsilon(\sum_{i=1}^3 \alpha_i - 2\alpha_j)}$$

For N>2 (Toda) there is ***NO formula for 3 generic primaries!***

The state of the art: [Fateev,Litvinov 2005-2008]

After ***specialisation*** *of one of the 3 primaries:*
$C(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)$ *can be written in terms of* ***known functions****!*

Primary with null vector at level 1

$$C(N\varkappa\omega_{N-1},\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{\left(2\mathcal{Q}-\sum_{i=1}^3 \boldsymbol{\alpha}_i,\rho\right)}{b}} \times$$
$$\times \frac{\Upsilon'(0)^{N-1}\Upsilon(N\varkappa)\prod_{e>0}\Upsilon((\mathcal{Q}-\boldsymbol{\alpha}_2,e))\Upsilon((\mathcal{Q}-\boldsymbol{\alpha}_3,e))}{\prod_{i,j=1}^N \Upsilon(\varkappa + (\boldsymbol{\alpha}_2-\mathcal{Q},h_i) + (\boldsymbol{\alpha}_3-\mathcal{Q},h_j))}$$

q-CFT

$$\Upsilon_q(x|\epsilon_1,\epsilon_2) = (1-q)^{-\frac{1}{\epsilon_1\epsilon_2}\left(x-\frac{\epsilon_+}{2}\right)^2} \prod_{n_1,n_2=0}^{\infty} \frac{(1-q^{x+n_1\epsilon_1+n_2\epsilon_2})(1-q^{\epsilon_+-x+n_1\epsilon_1+n_2\epsilon_2})}{(1-q^{\epsilon_+/2+n_1\epsilon_1+n_2\epsilon_2})^2}$$

$$= (1-q)^{-\frac{1}{\epsilon_1\epsilon_2}\left(x-\frac{\epsilon_+}{2}\right)^2} \left| \frac{\mathcal{M}(e^{-Rx})}{\mathcal{M}(e^{R\epsilon_+/2})} \right|^2$$

No Lagrangian, we only know the q-deformed symmetry algebra and representation theory!

[Shiraishi,Kubo,Awata,Odake][Frenkel,Reshetikhin][Lukyanov,Pugai][Feigin,Frenkel][Awata,Odake,Shiraishi,Kubo][Jimbo,Miwa]
[Awata,Yamada] [Mironov,Morozov,Shakirov,Smirnov]…

For N=2 (q-Liouville) q-DOZZ: [Nieri,Pasquetti,Passerini]

Bootstrap! Solve crossing equation!

For N>2 (q-Toda) can q-deform Fateev, Litvinov: [Mitev, EP]

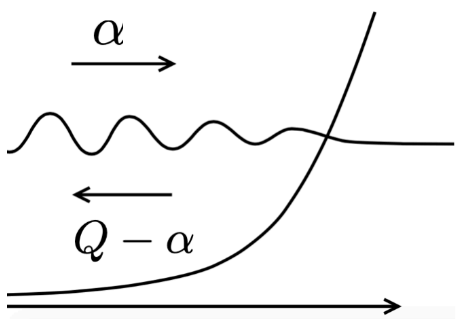
Primary with null vector at level 1

$$C_q(N\varkappa\omega_{N-1},\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) \cong \left(\frac{(1-q^b)^2(1-q^{b^{-1}})^{2b^2}}{(1-q)^{2(1+b^2)}}\right)^{\frac{\left(2\mathcal{Q}-\sum_{i=1}^3\boldsymbol{\alpha}_i,\rho\right)}{b}} \times \frac{\Upsilon_q'(0)^{N-1}\Upsilon_q(N\varkappa)\prod_{e>0}\Upsilon_q((\mathcal{Q}-\boldsymbol{\alpha}_2,e))\Upsilon_q((\mathcal{Q}-\boldsymbol{\alpha}_3,e))}{\prod_{i,j=1}^N\Upsilon_q(\varkappa+(\boldsymbol{\alpha}_2-\mathcal{Q},h_i)+(\boldsymbol{\alpha}_3-\mathcal{Q},h_j))}$$

Weyl Reflections and symmetry enhancement

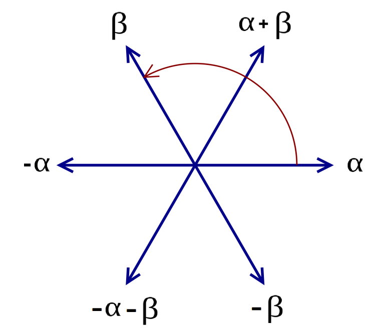
*Both T_N and the Toda 3pt function have **SU(N)³** symmetry.*

***Weyl reflections** imply **symmetry enhancement** for Toda 3pt functions.*
Which is the same as the symmetry enhancement of the T_N theory.



$$V_{\boldsymbol{\alpha}} = e^{(\boldsymbol{\alpha},\varphi)}$$

$$V_{w\circ\boldsymbol{\alpha}} = R^{w}(\boldsymbol{\alpha})V_{\boldsymbol{\alpha}}$$



$$C(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) = C^{\text{inv}}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)C^{\text{cov}}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)$$

[Fateev,Litvinov]

- *For N=2 (Liouville):* $C^{\text{cov}}(\alpha_1,\alpha_2,\alpha_3) = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{Q-\sum_{i=1}^{3}\alpha_i}{b}} \prod_{i=1}^{3}\Upsilon(2\alpha_i)$

$$C^{\text{inv}} = \left[\prod_{i=1}^{4}\Upsilon(\frac{Q}{2}+u_i)\right]^{-1} = \left[\prod_{i=1}^{4}G(\frac{Q}{2}+u_i)G(\frac{Q}{2}-u_i)\right]^{-1}, \quad \sum_{i=1}^{4}u_i = 0$$

*has **SO(8)** like T_2 !*

- *For N=3* C^{inv} *has the **E_6 symmetry** of T_3 !* [Argyres,Seiberg]

Our proposal

[Mitev,EP]

The Weyl covariant part is mapped to the extra d.o.f.

$$C_q^{\text{cov}}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) \simeq |\mathcal{Z}_{\text{extra}}|^2$$

$$m_i = (\boldsymbol{\alpha}_1 - \mathcal{Q}, h_i)$$

$$n_i = -(\boldsymbol{\alpha}_2 - \mathcal{Q}, h_i)$$

$$l_i = -(\boldsymbol{\alpha}_3 - \mathcal{Q}, h_{N+1-i})$$

- ☑ symmetries, zeros
- ☑ N=2 Liouville case

The Weyl invariant part to the 5D partition function

$$C_q^{\text{inv}}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) \simeq \mathcal{Z}_{T_N}^{S^4 \times S^1}$$

A first check

[Isachenkov,Mitev,EP]

- Take our proposed formula
- **Specialise** one of the three primaries: with null vector at level 1
- Explicit calculation is possible: **do the integrals and the sums**:

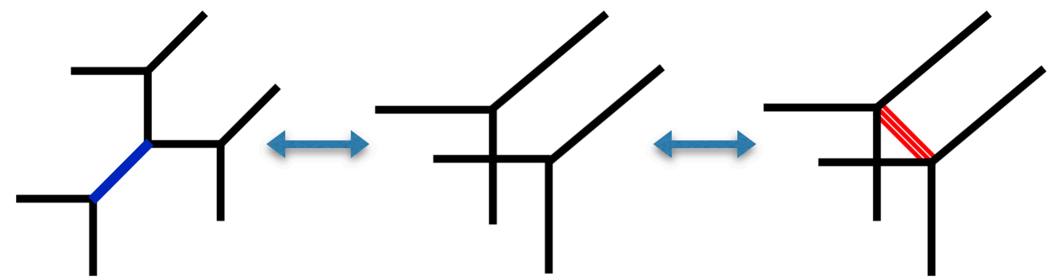
$$C_q^{\text{our}}\left(N\varkappa\omega_{N-1},\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3\right)=C_q^{\text{F.L.}}\left(N\varkappa\omega_{N-1},\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3\right)$$

$$C_q^{\text{inv}}\left(N\varkappa\omega_{N-1},\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3\right)\simeq\mathcal{Z}^{S^4\times S^1}_{\text{free hypers}}$$

From Topological string to Free fields

Geometric transitions: *Exchange the sums for products!* ***Take 4D limit!***

[Cheng, Dijkgraaf, Vafa]
[Aganagic, Haouzi, Kozcaz, Shakirov]



$$n - m - \ell = \underline{s}\epsilon_2 + \frac{\epsilon_1 + \epsilon_2}{2}$$

s = integer

sum = residues of integrals

$$\mathcal{Z}_2^{\text{top}} \propto \mathcal{I}_2 \xrightarrow{q \to 1} \int \prod_{i=1}^{s} dy_i \, y_i^{2b\alpha_1} (1 - y_i)^{2b\alpha_2} \prod_{i<j} (y_j - y_i)^{2b^2}$$

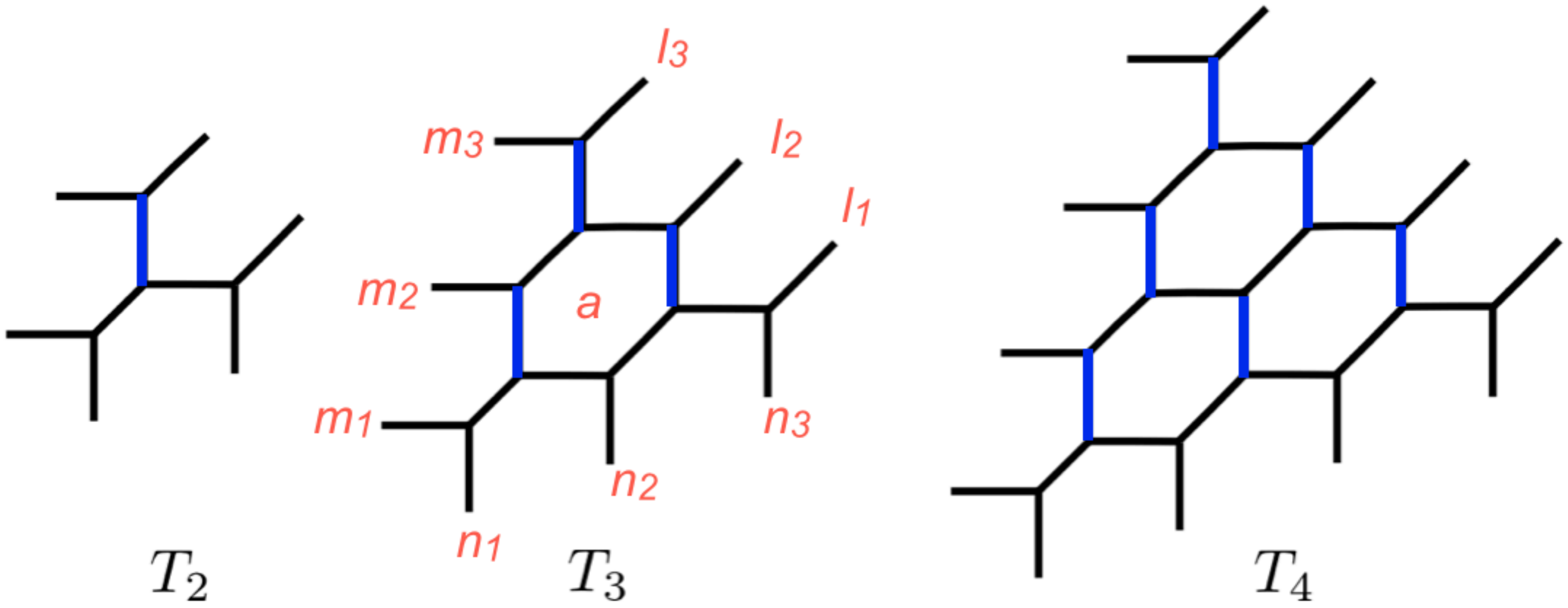
q-Selberg   Dotsenko-Fateev Free field rep.

From Z_{top} to free field integral: a change of the integration contour:
Jump when passing from one chamber in the parameter space to another.

[Coman,EP,Teschner]

From Topological string to Free fields

[Coman,EP,Teschner]



For N=3
3 integers: s_1, s_2, s_{12}

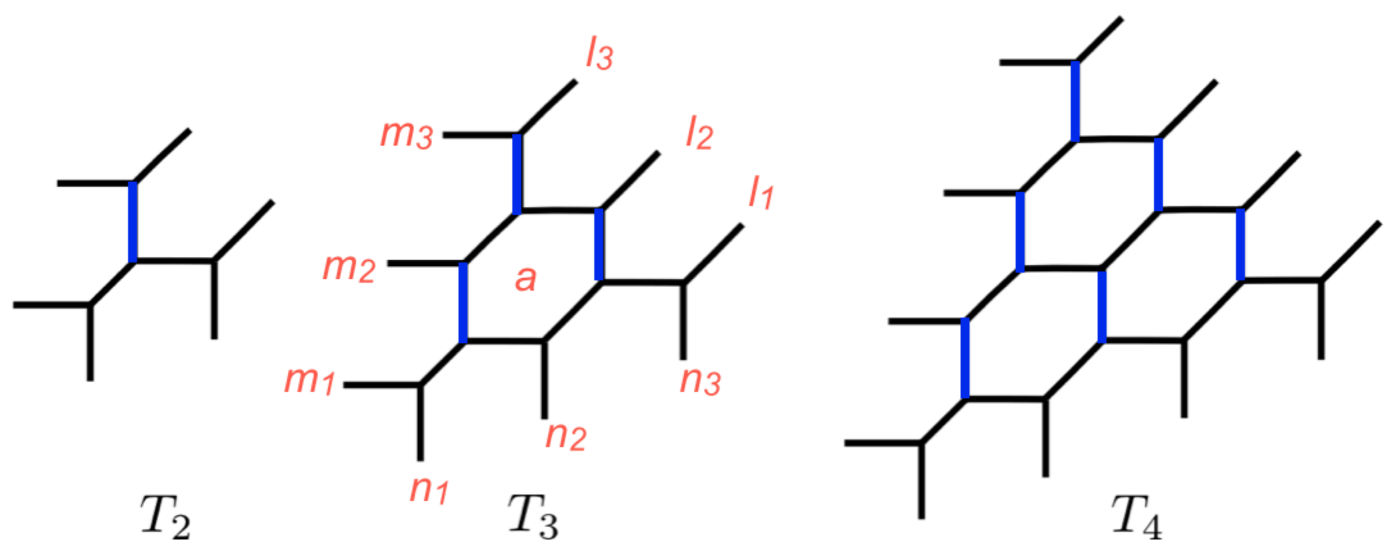
Z_{top} gives the integrand of the q-def. A_{N-1} generalisation of Selberg *

- Is the correct free field representation (**poles of 3pt function**).
- Integral representation is good for $q \to 1$ (4D) limit.

*We need to change contours to go from Z_{top} to the Selberg integral!

From Topological string to Free fields

[Coman,EP,Teschner]



For N=3
3 integers: s_1, s_2, s_{12}

- *The 3(N-1) masses obey screening:*

$$\boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + b \sum_{k=1}^{N-1} s_k e_k$$

N-1 screening charges associated to simple roots.

- *The (N-1)(N-2)/2 Coulomb moduli correspond to extra,* ***composite screening charges:***

$$a \propto s_{12}$$

We can define a **basis** for the space of conformal blocks on $\mathcal{C}_{0,3}$ given a normal ordering of screening charges and a contour prescription that is labelled by these (N-1)(N-2)/2 parameters.

The Coulomb moduli label the space of conformal blocks!

Conclusions and Outlook

- ☑ *Partition functions for the 5D T_N theories (**all parameters generic**).*
- ☑ *Proposal for the 3pt functions of q-Toda: **all 3 primaries generic**.*
- ☑ ***Important Check:** Geometric transitions: **free field representation**!******
- ☑ *Can **explicitly take the 4D limit** using the free field representation!* *****
- ☑ *Coulomb moduli **label the space of conformal blocks!***
- ☐ *For N>2 need to change basis for the contours.*
- ☑ *Gluing two T_2's: Instanton counting for **trifundamental half-hypers**.*
- ☐ *W-descendants, gluing 3-pt functions to get 4-pt functions.*

*For N>2 contours: work in progress

Thank you!