Extremal Correlators in Four Dimensions

Zohar Komargodski

Simons Center for Geometry and Physics, Stony Brook, NY

with Alba Grassi and Luigi Tizzano [**In Preparation**]
Previous Works with Efrat Gerchkovitz, Jaume Gomis, Po-Shen Hsin, Nafiz Ishtiaque, Avner Karasik, Silviu Pufu, Adam Schwimmer, Nathan Seiberg, Stefan Theisen.

Extremal Correlators

Integrability

Random Matrix Theory

Bootstrap

Supersymmetric gauge theories have provided us with beautiful examples of nontrivial phenomena in four-dimensional gauge theories:

- Electric-Magnetic Duality
- Spontaneous Symmetry Breaking
- Confinement Mechanisms

Our aim here is to continue this exploration into new directions:

- Properties of perturbation theory, instantons, resurgence....
- The physics of states very far from the vacuum, random matrix theory, a new expansion parameter...

The setup for this talk is:

Rank 1 $\mathcal{N} = 2$ SuperConformal Field Theories in 4d.

Some examples

- The free $\mathcal{N} = 2$ $U(1)$ vector multiplet.
- $\mathcal{N} = 4$ maximally supersymmetric Yang-Mills theory with gauge group $SU(2)$.
- $\mathcal{N} = 2$ gauge theory with gauge group $SU(2)$ and $N_F = 4$ (Seiberg-Witten theory).
- Rank 1 Argyres-Douglas theories.

The moduli space of vacua and the spectrum of chiral operators is very well understood in these theories. We will focus on the so-called Coulomb Branch operators:

$$\bar{Q}^1\mathcal{O} = 0 \ , \quad \bar{Q}^2\mathcal{O} = 0 \ ,$$

with $\mathcal{O}$ a superconformal primary.
For such operators $\Delta = R/2$, with R the $U(1)_R$ charge. These operators are singlets of $SU(2)_R$ and of the Lorentz group.

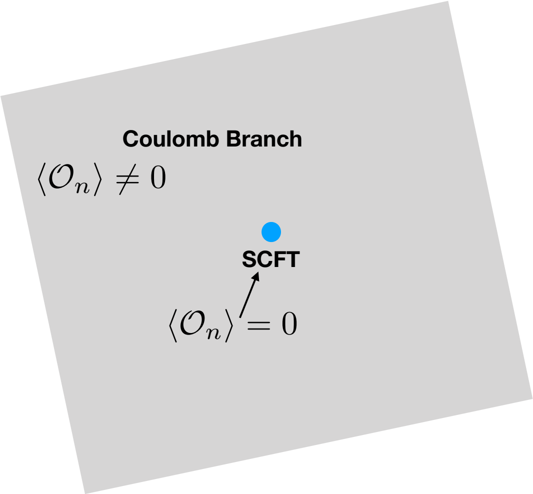
In the rank 1 examples of interest to us, these operators form a freely generated ring

$$\mathcal{O}_n \mathcal{O}_m \sim \mathcal{O}_{n+m} + \text{regular} \ , \quad n, m = 0, 1...$$

where $\mathcal{O}_0 \equiv 1$.

These operators are very important since they are order parameters for the Coulomb Branch vacua

$$\langle \mathcal{O}_n \rangle_{Coulomb-Branch} \neq 0$$


Coulomb Branch
$\langle \mathcal{O}_n \rangle \neq 0$
SCFT
$\langle \mathcal{O}_n \rangle = 0$

What about correlation functions of $\mathcal{O}_n$ at the fixed point itself? Clearly, purely chiral correlation functions vanish

$$\langle \mathcal{O}_{i_1}(x_1) \cdots \mathcal{O}_{i_n}(x_n) \rangle_{\mathbb{R}^4} = 0$$

Instead, we consider

$$\langle \mathcal{O}_{i_1}(x_1) \cdots \mathcal{O}_{i_n}(x_n) \bar{O}_J(y) \rangle_{\mathbb{R}^4} \ .$$

These are non-vanishing when $J = \sum_{k=1}^{n} i_k$. These are the "minimal" nontrivial correlation functions in the superconformal field theory. We refer to them as "extremal correlators."

- The space-time dependence is completely fixed [Papadodimas]:

$$\langle \mathcal{O}_{i_1}(x_1)\cdots\mathcal{O}_{i_n}(x_n)\bar{\mathcal{O}}_J(y)\rangle_{\mathbb{R}^4} = G_{2J}(\tau,\bar\tau)\prod_k \frac{1}{(y-x_k)^{2\Delta(\mathcal{O}_{i_k})}} \ .$$

- $G_{2J}(\tau,\bar\tau)$ is a function of the coupling constant and theta angle. It is perhaps the simplest nontrivial correlation function in four dimensional theories.
- Using the OPE we can reduce the computation to

$$\langle \mathcal{O}_J(x)\bar{\mathcal{O}}_J(y)\rangle_{\mathbb{R}^4} = G_{2J}(\tau,\bar\tau)\frac{1}{(x-y)^{2J\Delta_1}} \ ,$$

$$\langle \mathcal{O}_J(0)\bar{\mathcal{O}}_J(\infty)\rangle_{\mathbb{R}^4} = G_{2J}(\tau,\bar\tau) \ .$$

Some interesting questions about the G_{2J}:

- What are the properties of the perturbative contributions? Do they obey the Dyson rules and form an asymptotic series?
- An interesting limit is $J \to \infty$ which can be thought of as a state with finite charge density. Does perturbation theory commute with this limit? what is the physics of these heavy excitations?

The "Trivial" Examples:

- For the free vector multiplet theory we have $O_1 = \phi$ and hence

$$G_{2J} = \langle \phi^J(0)\bar{\phi}^J(\infty)\rangle_{\mathbb{R}^4} = J!\ .$$

- For $\mathcal{N} = 4$ theory there is a non-renormalization theorem [Lee-Minwalla-Rangamani-Seiberg]. One finds for gauge group $SU(2)$,

$$G_{2J} = \langle Tr(\phi^2)^J(0)\, Tr(\bar{\phi}^2)^J(\infty)\rangle_{\mathbb{R}^4} = (2J+1)!\ .$$

- For $SU(2)$ gauge theory with $N_f = 4$ it is easy to check that the correlator receives perturbative corrections, for instance, the first few terms are

$$G_2 = \frac{3}{8(Im\ \tau)^2} - \frac{135\zeta(3)}{32\pi^2}\frac{1}{(Im\ \tau)^4} + \frac{1575\zeta(5)}{64\pi^3}\frac{1}{(Im\ \tau)^5} + \cdots$$

(We use the usual notation $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$.)

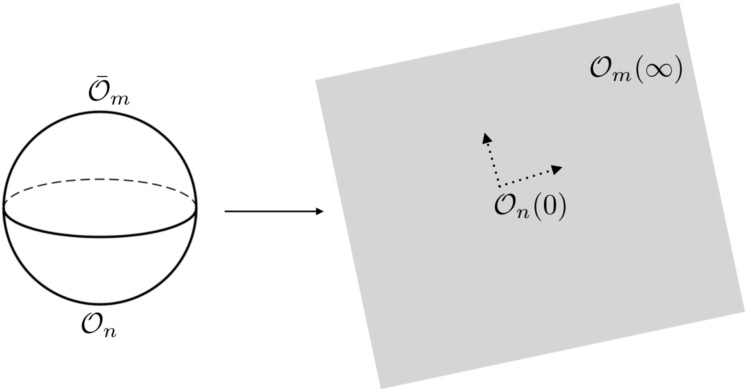
We are interested in a correlation function of conformal primary operators in $\mathbb{R}^4$. We would like to extract them from the S^4 partition function

$$Z_{S^4}[\tau, \bar{\tau}] = \int da\ a^2\ F(a; \tau, \bar{\tau})$$

where the partition function is given by an integral over the Coulomb Branch of some function $F(a; \tau, \bar{\tau})$ which was determined by Pestun in terms of the Nekrasov partition function ($F(a; \tau, \bar{\tau}) = |Z_{NEK}(a, \tau)|^2$).

Derivatives with respect to τ bring down insertions of the chiral ring generator $\mathcal{O}_1$ at the south pole of the sphere and derivatives with respect to $\bar{\tau}$ bring down insertions of $\bar{\mathcal{O}}_1$ at the north pole. Therefore

$$\langle \mathcal{O}_n(SP)\bar{\mathcal{O}}_m(NP)\rangle_{S^4} = \frac{1}{Z_{S^4}}\frac{\partial^n}{\partial\tau^n}\frac{\partial^m}{\partial\bar{\tau}^m}Z_{S^4}$$



$\bar{\mathcal{O}}_m$
$\mathcal{O}_n$
$\mathcal{O}_m(\infty)$
$\mathcal{O}_n(0)$

The mapping from S^4 to $\mathbb{R}^4$ turns out to be far more complicated because of operator mixing. This is due to contact terms and generalized trace anomalies.

For recent progress on a systematic understanding of these contact terms and trace anomalies see [Schwimmer, Theisen; Nakayama].

The same phenomenon occurs in two dimensions [Chen; Ishtiaque] and in three dimensions [Dedushenko, Fan, Pufu, Yacoby].

After the dust settles the answer for the G_{2J} is as follows:

Define the matrix

$$\mathcal{M}_{n,m} = \frac{1}{Z_{S^4}} \frac{\partial^n}{\partial \tau^n} \frac{\partial^m}{\partial \bar{\tau}^m} Z_{S^4} \ ,$$

then

$$G_{2J} = \frac{\det_{(J+1)\times(J+1)} \mathcal{M}}{\det_{J\times J} \mathcal{M}}$$

- E.g. for $\mathcal{N}=4$ one finds that $\mathcal{M}_{n,m}=\Gamma\left(m+n+\frac{3}{2}\right)$. The ratio of determinants can be computed exactly and one finds $(2n+1)!$ up to some unimportant normalization.
- Specializing $G_{2J}=\frac{\det_{(J+1)\times(J+1)}\mathcal{M}}{\det_{J\times J}\mathcal{M}}$ to $J=1$ we find that $G_2=\partial_\tau\partial_{\bar\tau}\log Z_{S^4}$. Since G_2 is the Zamolodchikov metric (which is Kähler) it follows that

$$Z_{S^4}=e^K\ .$$

This gives a nice interpretation for the four-sphere partition function!

One can understand $Z_{S^4} = e^K$ from a super-Liouville type theory in four dimensions (which is the same as a certain generalized trace anomaly).

Surprisingly, some of these claims hold even in non-conformal theories [Billo, Fucito, Korchemsky, Lerda, Morales].

- Another general conclusion from $G_{2J} = \frac{\det_{(J+1)\times(J+1)} \mathcal{M}}{\det_{J\times J} \mathcal{M}}$ is that the $G_{2J}(\tau, \bar{\tau})$ satisfy a Toda wave equation. In terms of $q_J = \log G_{2J} + K$, we have

$$\Box q_J = e^{q_{J+1} - q_J} - e^{q_J - q_{J-1}} \ .$$

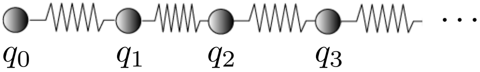
  The Toda oscillators have a boundary with $q_0 = K(\tau, \bar{\tau})$.

More generally, there is a mapping between extremal correlators and integrable systems, but the mapping is not very well understood beyond the rank 1 case.

We arrived at the Toda equation for extremal correlation functions

$$\Box q_J = e^{q_{J+1}-q_J} - e^{q_J - q_{J-1}}$$

from the trace anomalies that led to the ratio of determinants. But one can arrive at the equation directly by Ward identities in $\mathbb{R}^4$ through the four-dimensional tt* equations! [Papadodimas; Baggio-Niarchos-Papadodimas]



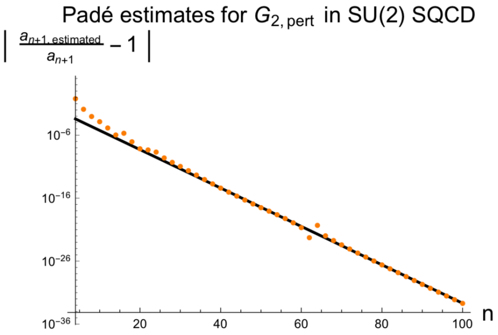
Let us study the partition function in the zero-instanton sector:

$$Z_{pert}[S^4][\tau,\bar{\tau}] = \int_{\mathbb{R}} da\, e^{-4\pi Im\tau a^2}(2a^2)\frac{H(2ia)H(-2ia)}{|H(-ia)H(ia)|^4}$$

with $H(x) \equiv G(1+x)G(1-x)$. We then construct the matrix $\mathcal{M}_{n,m}$, take derivatives, and extract the G_{2J}^{pert}.

$G_{2J}^{pert}(Im\tau)$ is a very rich object. First, the perturbative expansions in $Im\tau$ is an asymptotic expansion. It is Borel summable and it predicts the existence of instantons. All the poles of the Borel transform are on the negative Borel axis [Russo; Honda].

$G_{2J}^{pert}(Im\tau)$ satisfies QCD conjectures [Karliner 1998 for a review] about the Padé approximant being exponentially close to the right answer.


Padé estimates for $G_{2,\text{pert}}$ in SU(2) SQCD
$\left| \frac{a_{n+1,\text{estimated}}}{a_{n+1}} - 1 \right|$
10^{-6}
10^{-16}
10^{-26}
10^{-36}
20
40
60
80
100
n

Technically, the limit of large J looks hard. We need to compute the determinants of large matrices and the expansion in g_{YM}^2 unfortunately fails. If one insists on expanding in g_{YM} one finds terms such as $g_{YM}^2 J$ which are foreboding for the prospects of perturbation theory at large J. This is a general difficulty in QFT.

We can appeal to the following trick (Andréief):

To compute $\det_{J\times J}\mathcal{M}$ we can consider instead a $J \times J$ chiral random matrix model:

$$\int_0^\infty \prod_{i=1}^{J} dy_i \prod_{k<m} (y_k - y_m)^2 e^{\sum_{i=1}^{J} V(y_i)} ,$$

with V the potential of the eigenvalues. It is given by

$$V(x) = -4\pi Im\tau \; x + \frac{1}{2}\log x + \log\left(\frac{H(2i\sqrt{x})H(-2i\sqrt{x})}{|H(-i\sqrt{x})H(i\sqrt{x})|^4}\right)$$

Since the eigenvalues are positive definite, this is not the standard ensemble of Hermitian matrices. Rather, it is the Wishart-Laguerre chiral ensemble.

Our original theory was rank 1, but now we see that for the description of heavy states it is more convenient to use “emergent” random $J \times J$ matrices. It is tempting to think about these matrices as describing the quanta of which the heavy states are made.

Reminiscent of the connection between the Eigenstate Thermalization Hypothesis and Random Matrix Theory.

We can learn many things from the RMT. First, let us take $J \to \infty$ with *fixed Im* τ. If we truncate the potential as $V(x) = -4\pi Im\tau\ x + \log x$ then the RMT is exactly solvable. The eigenvalue are distributed according to Marchenko-Pastur

$$\rho(x) \sim \frac{1}{N}\frac{1}{\sqrt{x/N}}\sqrt{\frac{1}{4\pi Im\ \tau} - \frac{x}{N}}\ ,$$

As a result the vast majority of eigenvalues are of order N and this justifies self-consistently the truncation we made.

We can therefore extract the few leading pieces in J for the extremal correlators from the simplified random matrix model

$$\int_0^\infty \prod_{i=1}^{J} dy_i \prod_{k<m} (y_k - y_m)^2 e^{-4\pi Im\tau \; y_i + \log y_i} \; .$$

Taking the ratio of the determinants gives

$$G_{2J} = 2J \log J + 2 \log J - 2J \log Im \; \tau + \cdots$$

Two important comments about

$$G_{2J} = 2J \log J + 2 \log J - 2J \log Im\ \tau + \cdots$$

- At sub-leading orders in J, the eigenvalues near the origin (the "hard edge") become important and we cannot trust our approximation beyond the order above.
- the dependence on $Im\ \tau$ above is through a term linear in J. This means it can be removed by a redefinition of the operators. So there is really no dependence on $Im\ \tau$ at very large J.

A very nice approach to the physics of large J was developed by [Hellerman, Maeda, Orlando, Reffert, and Watanabe]. They are predicting that at large J we have a theory of non-relativistic Goldstone modes related to the conformal axion-dilaton. The RMT approach agrees with their predictions and allows to go beyond effective field theory.

So far we made some comments about the limit of large J with fixed $Im\ \tau$. Since $Im\tau\ x \sim J$ for the typical eigenvalue x, it is natural to take

$$\frac{J}{Im\tau} = \lambda \ ,$$

where λ is fixed in the large J limit.

From the point of view of the random matrix model, it is the ordinary 't Hooft limit. We get a genus expansion (up to a $\log J$ that we omit)

$$\log G_{2J} = \sum J^{-m} C_m(\lambda)$$

We can readily solve the model in the planar limit since only the piece $V(x) = -4\pi Im\tau\ x$ contributes (the other terms are subleading). Thus $C_{-1} = 2\log\lambda$.

This double scaling limit is not tightly constrained by effective field theory. In a beautiful paper [Bourget, Rodriguez-Gomez,Russo] have done an explicit perturbative analysis and showed that the double scaling limit is obeyed! [Beccaria] extended those results to a very high order in perturbation theory.

$$C_0(\lambda) = -\frac{9\zeta(3)}{32\pi^4}\lambda^2 + \frac{25\zeta(5)}{128\pi^6}\lambda^3 + \cdots \ .$$

The RMT explains why such an expansion exists: it is the standard 't Hooft expansion for the emergent matrices! RMT allows us to perform the genus expansion efficiently and explore non-perturbative effects in the 't Hooft coupling.

Double scaling limit at large charge $\longleftrightarrow$ 't Hooft expansion ,

charge $J \longleftrightarrow$ N ,

$g_{YM}^2 J \longleftrightarrow$ 't Hooft coupling λ ,

Gauge instantons are exponentially small in both cases.

- This connection between extremal correlators, integrability, random matrix theory and a new 't Hooft like limit is completely general.
- This gives us a glimpse into the general properties of perturbation theory and also perturbation theory in highly excited states. We saw that a 't Hooft-like coupling emerges and one can study a new genus expansion.

Thank you !