# FROM RESURGENCE TO BPS STATES

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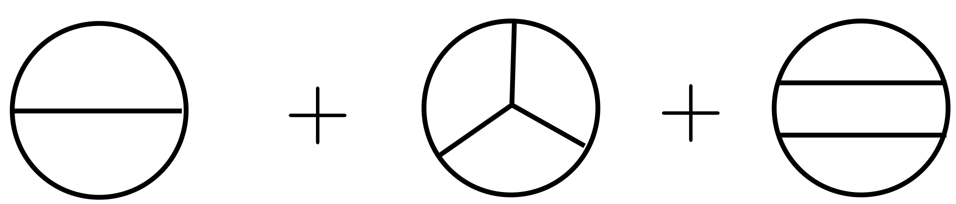
[Ito-M.M.-Shu, 1811.04812]
[M.M.-Reis,1905.09569&1905.09575]

+work in progress with Alba Grassi and Jie Gu

# Perturbative and non-perturbative sectors

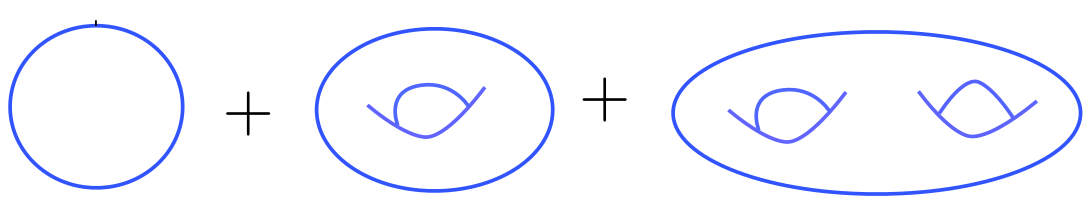
In quantum theories, observables are often computed by perturbative series in a small coupling constant. Sometimes we have to add corrections due to non-perturbative effects, which are also given by infinite series

instantons/condensates



$$+\cdots \quad + g^b \mathrm{e}^{-S/g^2}\left(1+\mathcal{O}(g)\right)$$

D-branes



$$+\cdots \quad + g_s^b \mathrm{e}^{-S/g_s}\left(1+\mathcal{O}(g_s)\right)$$

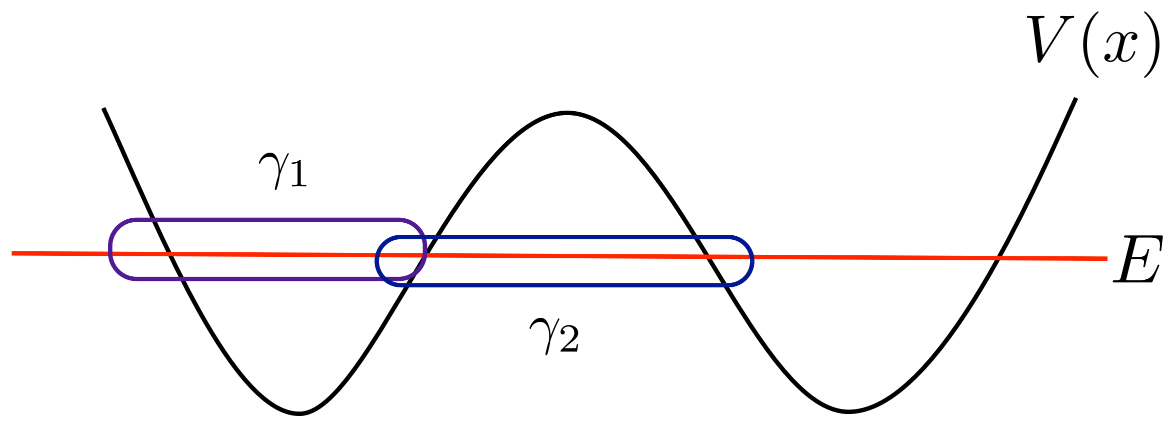
$$g_s^{-2}F_0 + F_1 + g_s^2 F_2 + \cdots$$

# A simple example: all-orders WKB

In the WKB method, perturbative and non-perturbative sectors can be understood geometrically

"WKB curve"

$$\Sigma(x,p) = H(x,p) - E = 0$$




quantum one-form

$$p(x)\mathrm{d}x \longrightarrow p(x,\hbar)\mathrm{d}x$$

quantum periods

$$\Pi_a(\hbar) = \oint_{\gamma_a} p(x,\hbar)\mathrm{d}x \sim \sum_{n\geq 0} \Pi_a^{(n)} \hbar^{2n}$$

perturbative: $\Pi_1(\hbar)$

non-perturbative (tunneling) $\mathrm{e}^{-\Pi_2(\hbar)/\hbar}$

It is well-known that all these series (in both the perturbative and the non-perturbative sectors) are asymptotic and do *not* define functions: their coefficients grow as

$$n! \quad \text{or} \quad (2n)!$$

Can we make sense of these series? What are the mathematical structures governing them? Could we use these structures to obtain exact results?

In this talk I will try to provide some answers to these questions.

# The Borel triangle

The Borel method is a systematic (and traditional) way of making sense of factorially divergent formal power series

$$\varphi(z) = \sum_{n \geq 0} a_n z^n$$

Borel transform

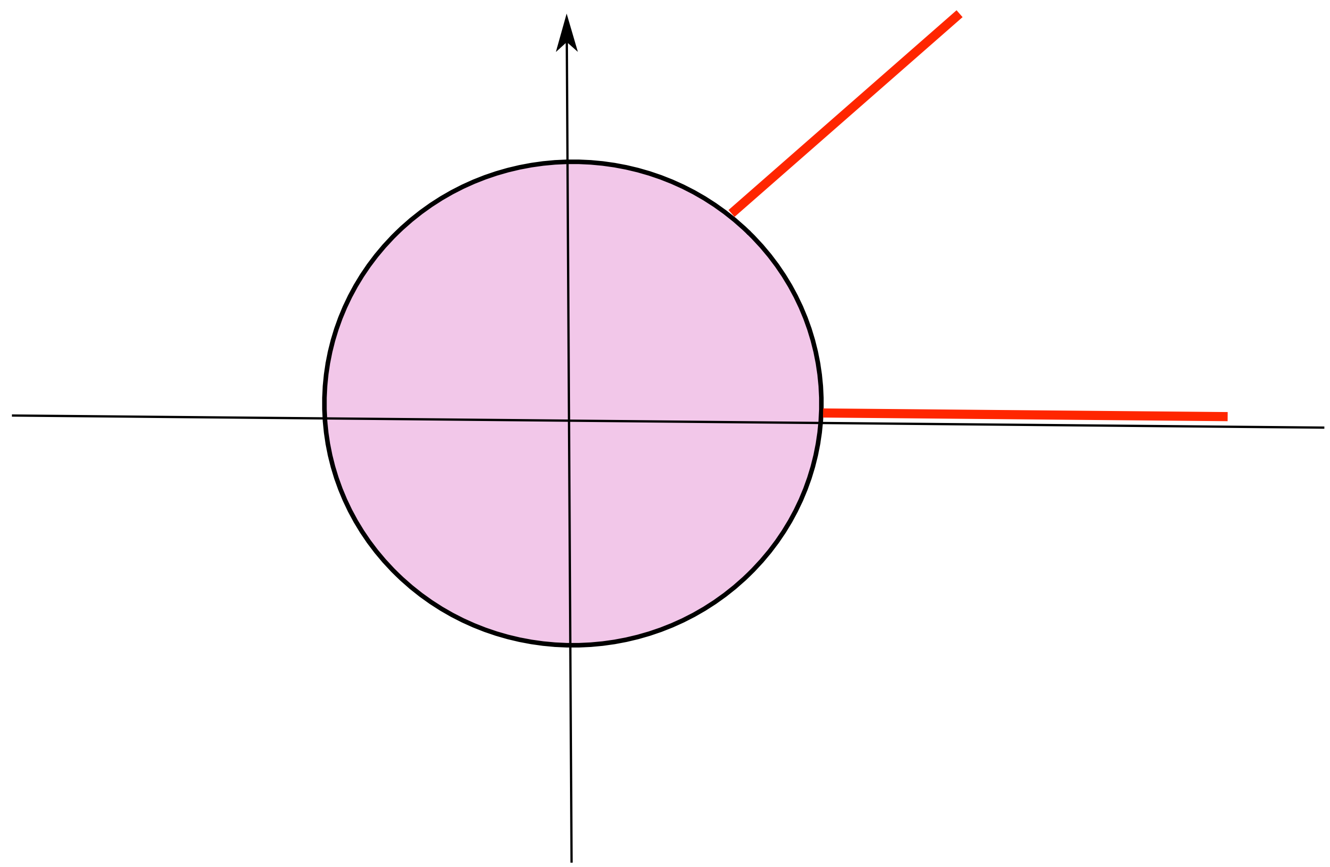
$$\widehat{\varphi}(\zeta) = \sum_{n \geq 0} \frac{a_n}{n!} \zeta^n$$

Laplace transform

$$s(\varphi)(z) = \int_0^\infty \mathrm{e}^{-\zeta} \widehat{\varphi}(z\zeta) \mathrm{d}\zeta$$

Borel resummation

The Borel transform $\widehat{\varphi}(\zeta)$ is analytic at the origin. Very often it can be analytically continued to the complex plane, displaying *singularities* (poles, branch cuts).

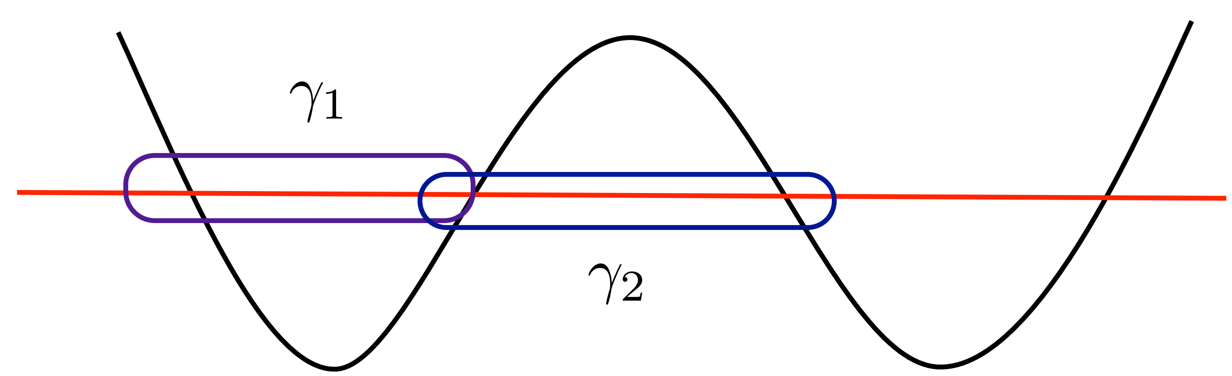


Singularities along the positive real line are obstructions to Borel resummation

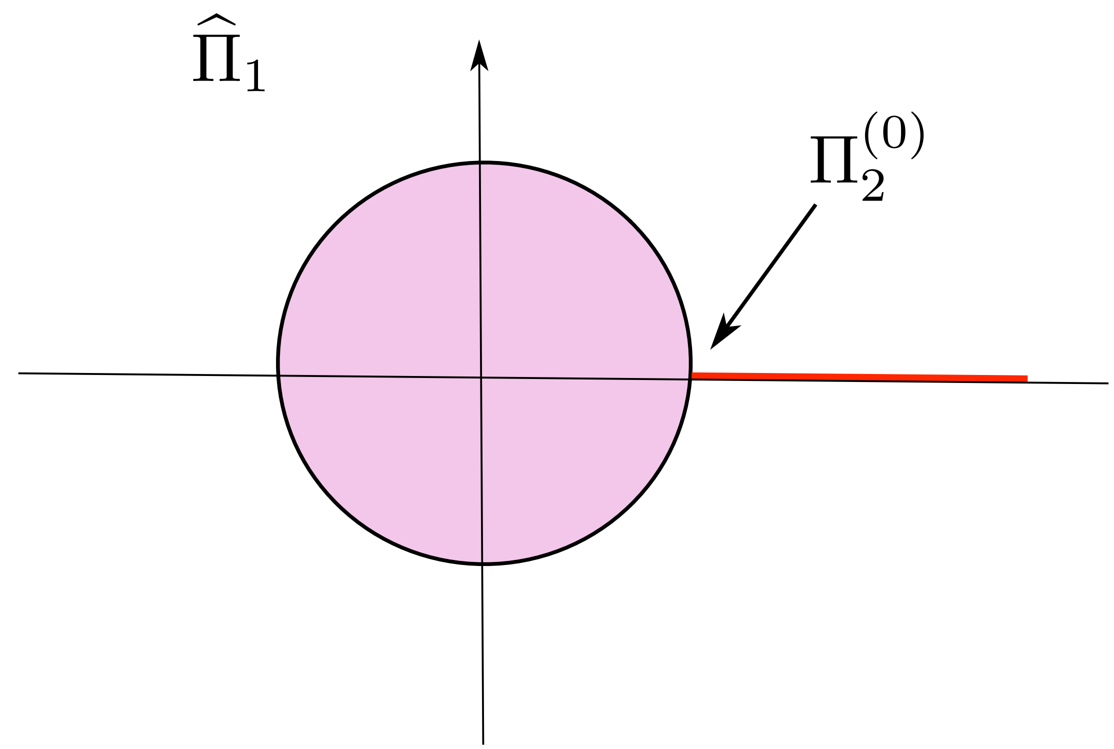
# Resurgence I: basics

Consider the series in a given sector and its Borel transform. Its singularities know about *other* sectors! This is the classical manifestation of **resurgence** and goes back to [Bender-Wu, 1973]

Example:
double well in QM




The first Borel singularity for the *perturbative* series is given by the (classical) *tunneling* amplitude



# An example in condensed matter

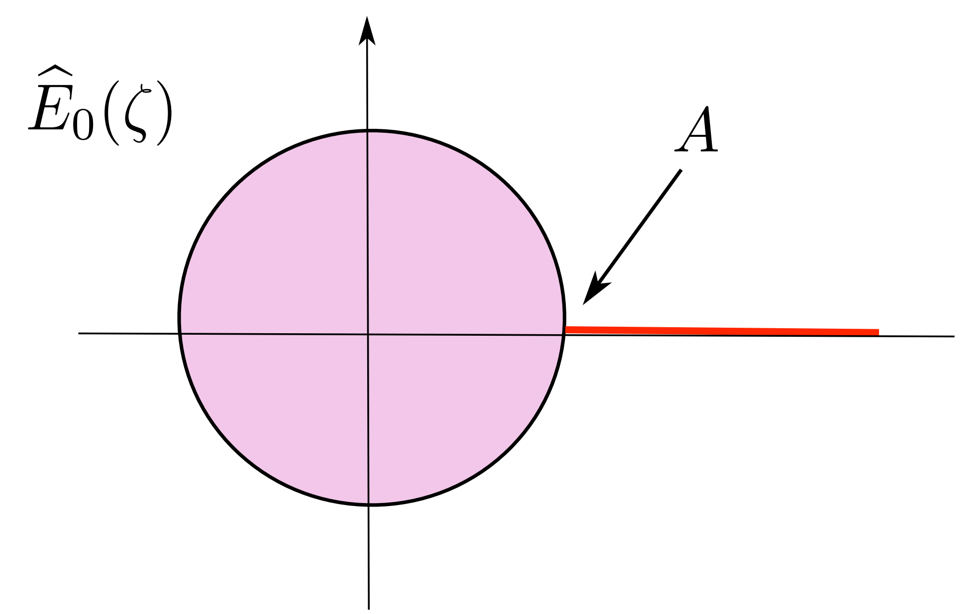
Consider a spin 1/2 Fermi gas with a weak attractive interaction

$$\mathcal{H} = -\sum_{\sigma=\uparrow,\downarrow} \bar{\psi}_\sigma \frac{\nabla^2}{2m} \psi_\sigma - g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow$$

ground state energy $\qquad E_0(g) \sim \sum_{n\geq 0} a_n g^n$

superconducting energy gap $\qquad \Delta^2 \sim \mathrm{e}^{-A/g}$

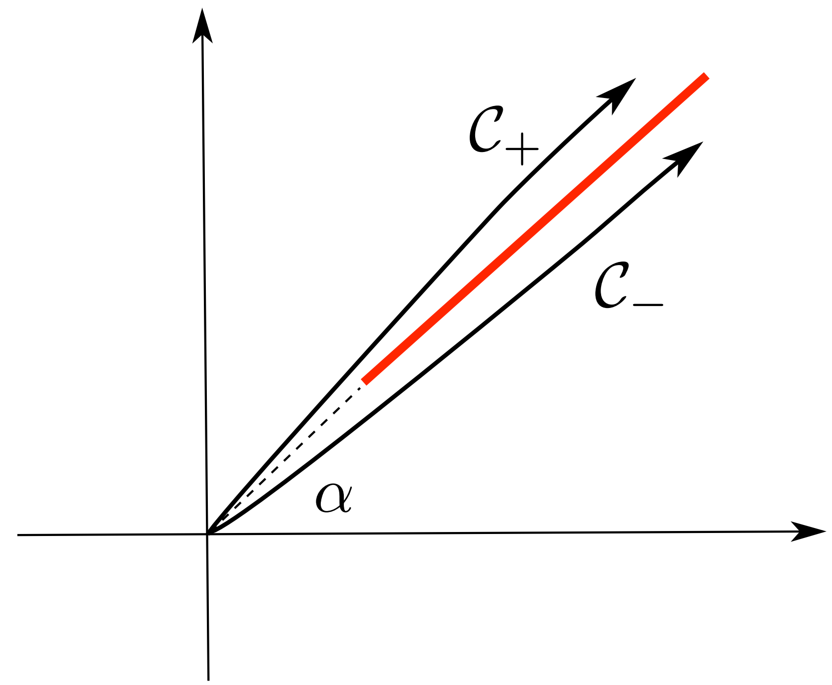
The first Borel singularity for the ground state energy is determined by the energy gap
[M.M.-Reis]



# Resurgence 2: more advanced tools

Don't be afraid of Borel singularities: do lateral resummations!

$$s_{\pm\alpha}(\varphi)(z) = \int_{\mathcal{C}_\pm} \mathrm{e}^{-\zeta} \widehat{\varphi}(z\zeta) \mathrm{d}\zeta$$




*Stokes discontinuity (or Stokes automorphism)*

$$\mathrm{disc}_\alpha(\varphi) = s_{+\alpha}(\varphi) - s_{-\alpha}(\varphi)$$

"Advanced" resurgence [Ecalle]: the Stokes discontinuity of a series in a given sector is a function of the series in other sectors (*and nothing else*).

# The analytic bootstrap

Suppose we have a quantum theory and we know (1) the Stokes discontinuities of the series in all sectors, and (2) their classical limit.

Can we then reconstruct the *exact (resummed)* series?

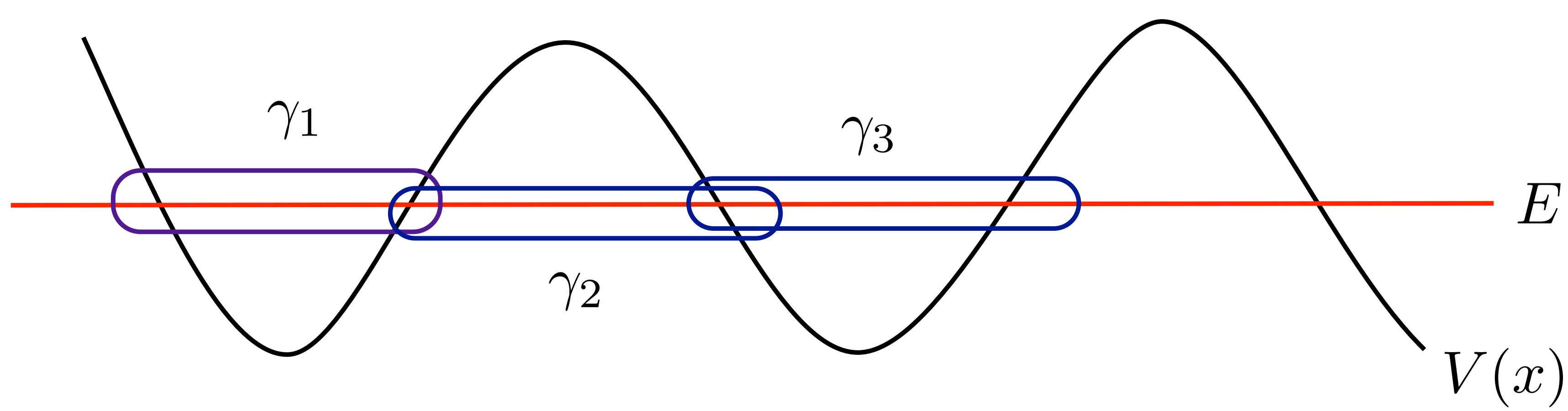
This approach to quantum theory was proposed by André Voros in 1983. He called it the *analytic bootstrap*. It is in fact a typical Riemann-Hilbert problem.

# A solvable example

There is at least one example in which the analytic bootstrap can be solved in an interesting way: the all-orders WKB method with polynomial potentials

$$H(x,p) = p^2 + V(x) \qquad V(x) = x^{r+1} - \sum_{i=1}^{r} u_i x^{r+1-i}$$

"minimal" chamber in moduli space




$\Pi_1 \qquad \Pi_2 \qquad \ldots$

The Stokes discontinuities in this case are given by the *Delabaere-Pham formula:*

$$\mathrm{disc}(\Pi_a) = \log\left(1+\mathrm{e}^{-s(\Pi_{a-1})/\hbar}\right) + \log\left(1+\mathrm{e}^{-s(\Pi_{a+1})/\hbar}\right)$$

+classical limit $\quad \Pi_a(\hbar) \sim \Pi_a^{(0)}, \qquad \hbar \to 0$

Building on [Gaiotto-Moore-Neitzke, Cecotti-Vafa], one solves this Riemann-Hilbert problem in terms of TBA-like equations [Ito-M.M. -Shu]

$$\epsilon_a(\theta) = \Pi_a^{(0)}\mathrm{e}^{\theta} - \int_{\mathbb{R}} \frac{L_{a-1}(\theta')}{\cosh(\theta-\theta')}\frac{\mathrm{d}\theta'}{2\pi} - \int_{\mathbb{R}} \frac{L_{a+1}(\theta')}{\cosh(\theta-\theta')}\frac{\mathrm{d}\theta'}{2\pi}$$

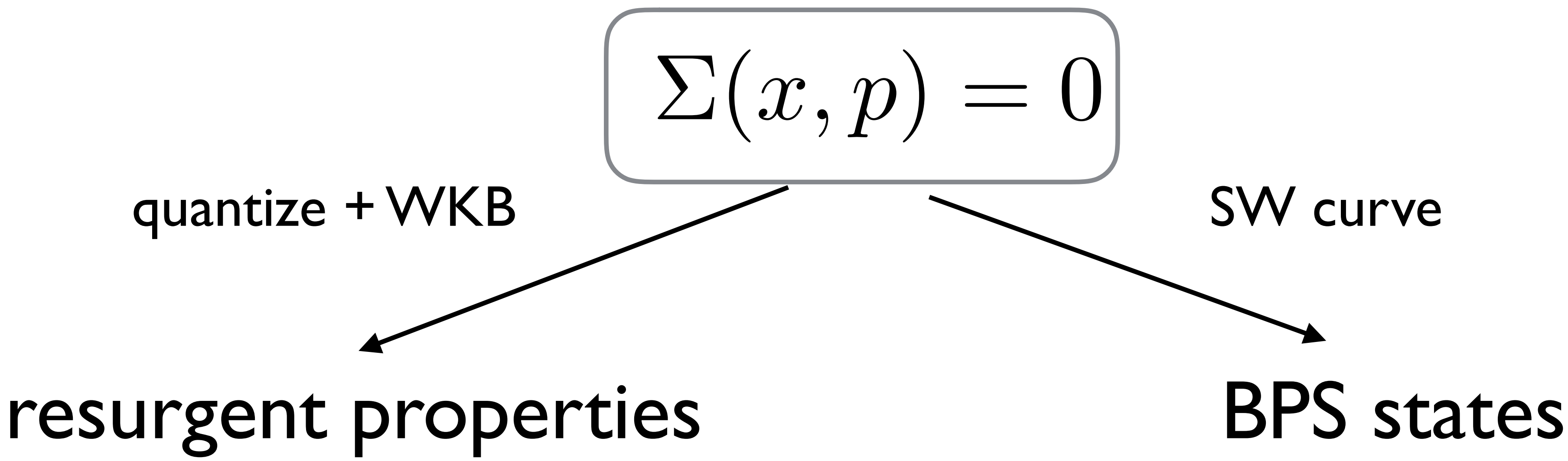
$$\hbar = \mathrm{e}^{-\theta} \qquad \epsilon_a(\theta) = \frac{1}{\hbar}s(\Pi_a)(\hbar) \qquad L_a(\theta) = \log\left(1+\mathrm{e}^{-\epsilon_a(\theta)}\right)$$

This provides a “resurgent” derivation of a conjecture by Gaiotto. It extends the *ODE/IM correspondence* of Dorey-Tateo (which was derived for monic potentials) to *arbitrary polynomial potentials*.

Combined with appropriate quantization conditions, the TBA equations provide powerful tools to solve spectral problems exactly [Ito-M.M.-Shu]

As we move in moduli space, one has to consider additional quantum periods [Delabaere-Dillinger-Pham, 1993] and include them in theTBA equations [Alday et al.,Toledo]. This is an example of the wall-crossing phenomenon.

# From resurgence to BPS states

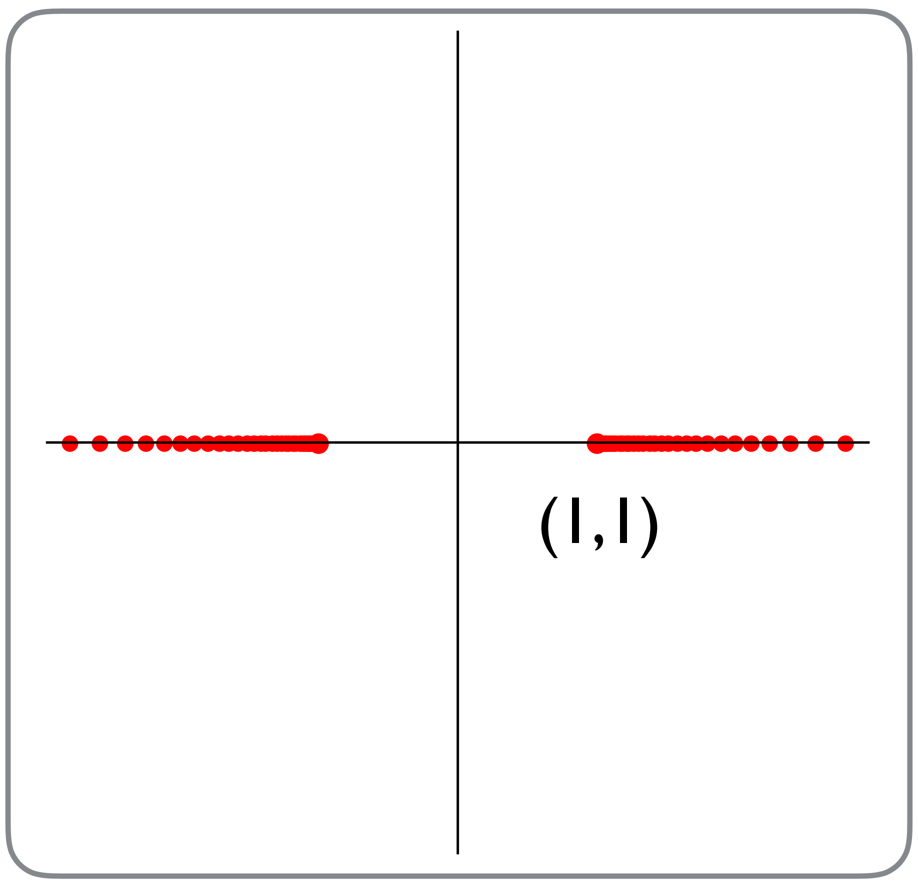



| Resurgent properties | BPS states |
|---|---|
| $\langle \gamma_a \rangle$ | $\Gamma$ |
| $\Pi_a^{(0)}$ | $Z(\gamma_a)$ |
| Borel singularities | BPS spectrum |
| Stokes discontinuities | KS morphisms |

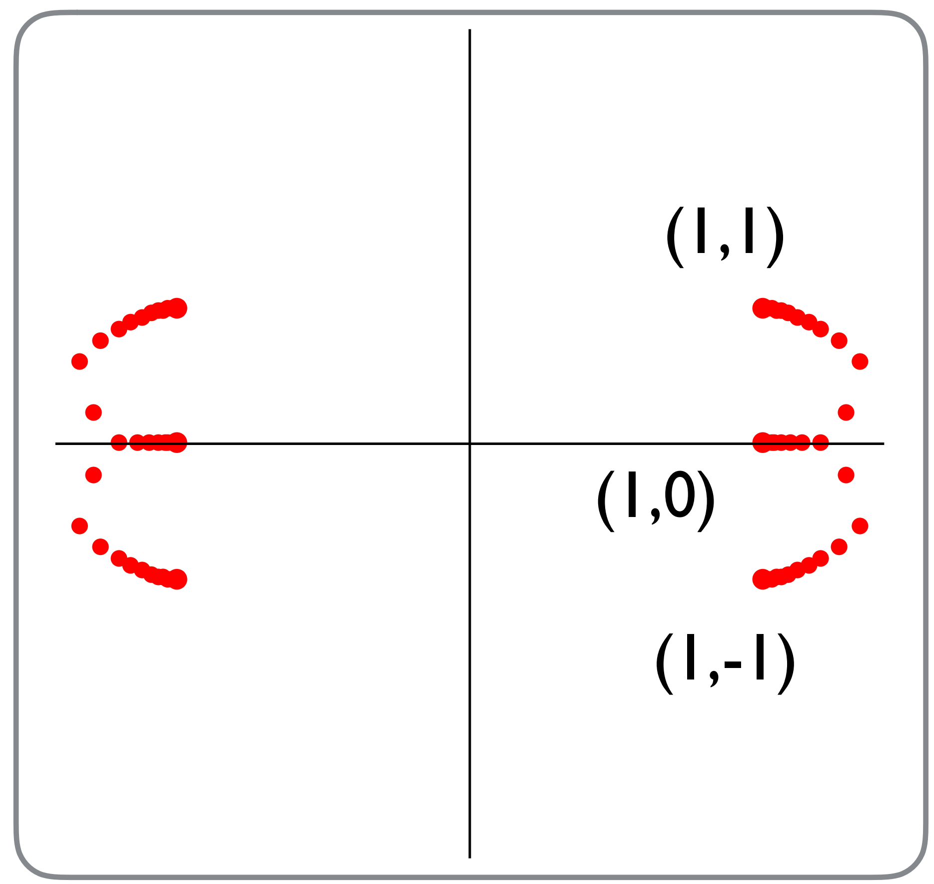
# Revisiting Seiberg-Witten theory

$$\Sigma(x,p) = 2\cosh(p) + x^2 - u = 0 \longrightarrow \begin{aligned} a(\hbar) &= \sum_{n\geq 0} a^{(n)}\hbar^{2n} \\ a_D(\hbar) &= \sum_{n\geq 0} a_D^{(n)}\hbar^{2n} \end{aligned}$$

Borel plane for the magnetic quantum period




strong coupling




weak coupling

# Examples to explore

mirror curve of local CY

resurgent properties of quantum mirror curve

spectrum of BPS states [Douglas, …][Eager et al., Banerjee et al.]

A-polynomial of a knot

resurgent properties of quantum invariants [Garoufalidis,Gukov-M.M.-Putrov]

BPS states?

# Conclusions and outlook

• Resurgence is a universal framework to understand the relation between perturbative and non-perturbative physics

•Most developed in “simple” cases (quantum mechanics) but increasing number of applications (e.g. condensed matter)

•Suggests a new approach to quantum theory -the *analytic bootstrap*-where the basic building blocks are Stokes morphisms. What is the algebraic structure of these morphisms? Can they be classified?

•Fascinating interplay with the theory of BPS states in SUSY gauge theories, with many potential applications.

# Thank you for your attention!