On non-planar scattering amplitudes

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In collaboration with
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Motivation

- analytic `data` for amplitudes essential
  catalyst for developing new methods
- N=4 super Yang-Mills perfect laboratory
  `QFT analog of hydrogen atom in QM`

- impressive progress, but mostly limited to the planar sector of the theory
- intriguing insights into properties of non-planar loop integrands
- very few analytic results for integrated answer

This talk: full 3-loop four-particle amplitude

loop integrands

many different representations available in literature

- form admitting BCJ duality

  [Bern, Carrasco, Dixon, Johansson, Roiban 2008]

- manifest UV properties

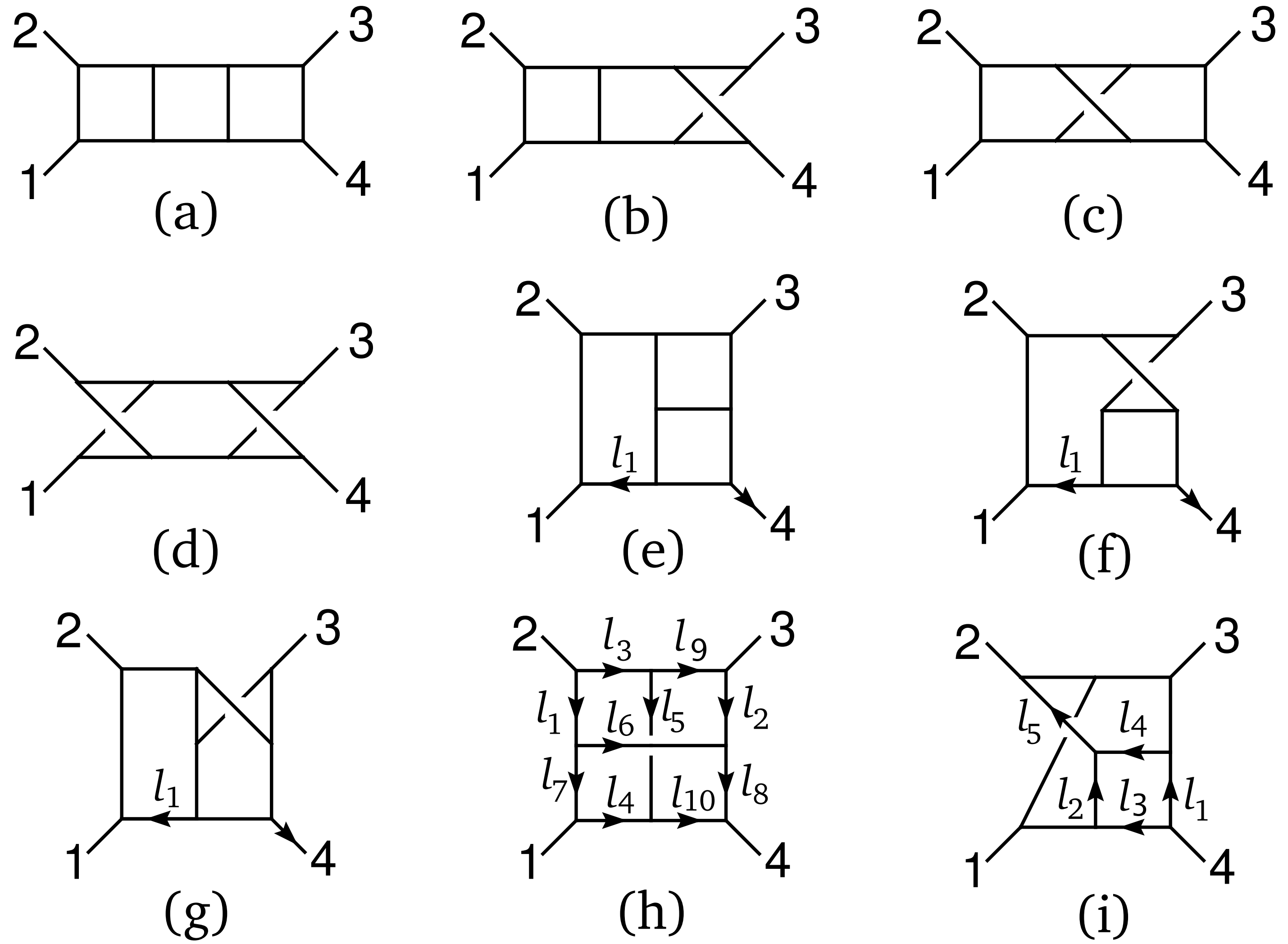
  [Bern, Carrasco, Dixon, Johansson, Roiban 2012]

- d-log forms, absence of `poles at infinity`

  [Arkani-Hamed, Bourjaily, Cachazo, Trnka 2014]
  [Bern, Herrmann, Litsey, Stankowicz, Trnka 2015+2016]

difficulty: non-planar 3-loop integrals unknown

3- loop integrals



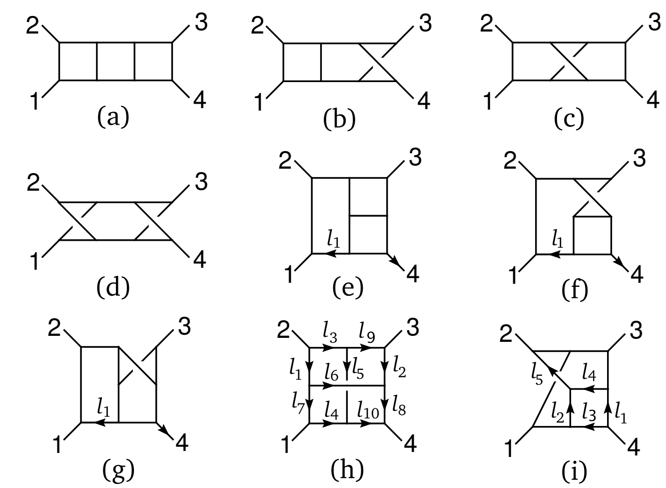
- all planar integrals of type (a),(e) computed in

  [JMH, A.V. Smirnov, 2013]

- non-planar sample integrals [JMH, A.V. Smirnov, V.A. Smirnov, 2013]
- extended calculation to all non-planar families

  [JMH, B. Mistlberger, A.V. Smirnov, to appear]

Main points of the method



- use integral basis that has unit leading singularities

  [Arkani-Hamed et al; JMH]

- differential equation method

  [Kotikov; Bern, Dixon, Kosower; Remiddi; Gehrmann; JMH 2013]

$$\partial_x \vec{f}(x;\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] \vec{f}(x;\epsilon) \qquad x = t/s$$

- boundary conditions from consistency and symmetry
- all constants expressed in terms of multiple zeta values
- solution in Laurent expansion

Integral reduction

- Feynman integrals satisfy integration-by-parts identities
- public codes available for solving them
- we used a private implementation by Bernhard Mistlberger

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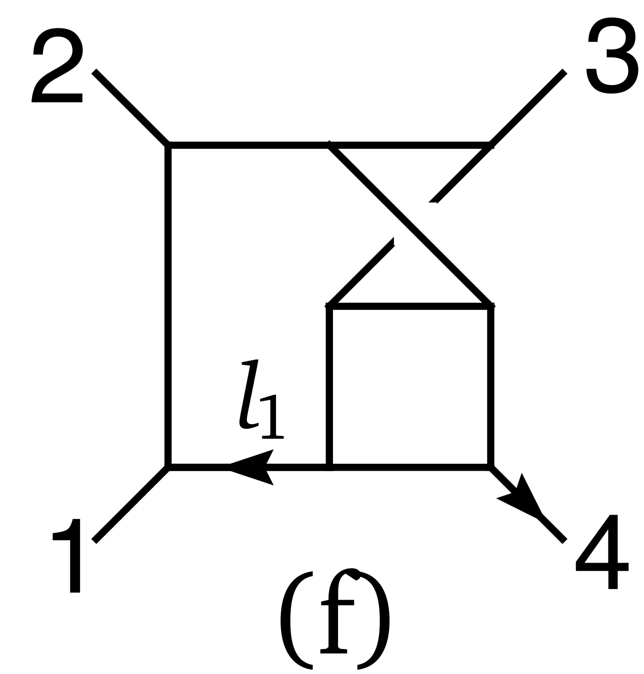
We got some additional help from u0001

Many thanks to computing department ZDV at Mainz university!

example

- unit leading singularities basis integral

$$I = s(s+t) I_f[(l_1 + p_4)^4]$$

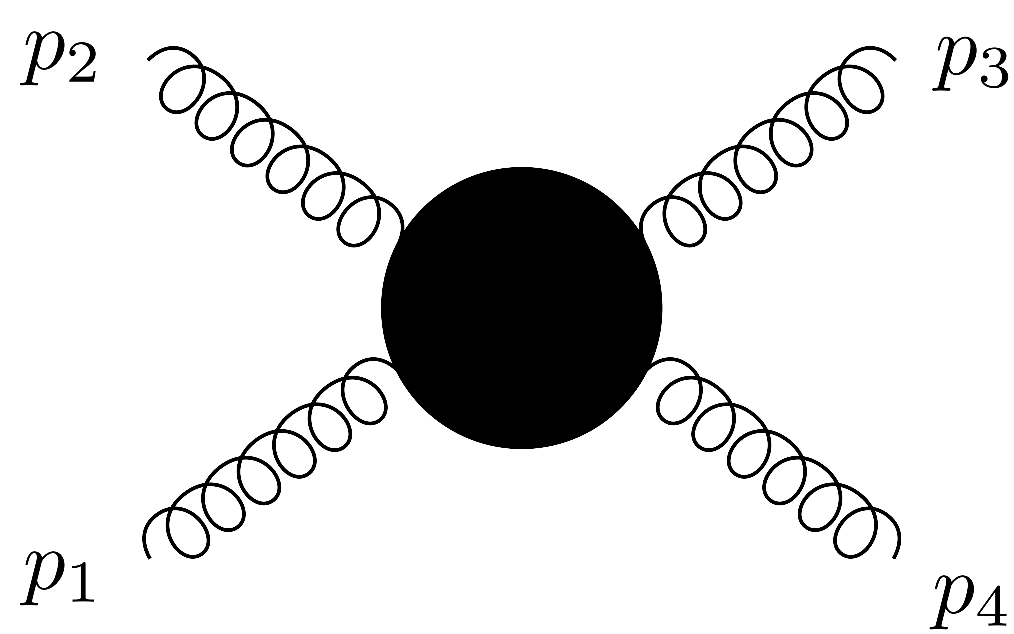


- epsilon expansion

$$\begin{aligned} I = \; & -\frac{1}{\epsilon^6}\frac{47}{36} \\ & +\frac{1}{\epsilon^5}\left[-\frac{8i\pi}{3} + \frac{8H_{-1}}{3} - \frac{3H_0}{4}\right] \\ & +\frac{1}{\epsilon^4}\left[-4H_{-1,-1} + H_{-1,0} + \frac{H_{0,0}}{4} + \frac{503\zeta_2}{24} + 4i\pi H_{-1} - i\pi H_0 + H_{-2}\right] \\ & +\frac{1}{\epsilon^3}\left[2i\pi H_{0,0} + 2H_{-2,-1} - 2H_{-2,0} - 2H_{-1,0,0} + \frac{21}{4}H_{0,0,0} + 31i\pi\zeta_2 \right. \\ & \qquad \left. + \frac{715\zeta_3}{36} - 2i\pi H_{-2} - 33\zeta_2 H_{-1} + \frac{355\zeta_2 H_0}{24} - 2H_{-3}\right] \\ & + \mathcal{O}(\epsilon^{-2}) \end{aligned}$$

- functions H: harmonic polylogarithms, uniform weight

Application to four-particle scattering amplitude



- Mandelstam variables

$$s = (p_1 + p_2)^2 \qquad t = (p_2 + p_3)^2$$

- dimension $D = 4 - 2\epsilon$

- expansion in coupling $\alpha = \frac{g^2}{4\pi^2}(4\pi e^{-\gamma_{\mathrm{E}}})^\epsilon$

$$\mathcal{A}(p_i;\epsilon) = \mathcal{K} \sum_{L=0}^{\infty} \alpha^L \mathcal{A}^{(L)}(s,t;\epsilon).$$

Color decomposition

- trace basis

$$tr(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) = tr(1234).$$

$$\begin{aligned} C_1 &= tr(1234) + tr(1432) & C_4 &= tr(12)tr(34) \\ C_2 &= tr(1243) + tr(1342) & C_5 &= tr(13)tr(24) \\ C_3 &= tr(1423) + tr(1324) & C_6 &= tr(14)tr(23) \end{aligned}$$

- expansion in powers of N

$$\mathcal{A}^{(L)} = \sum_{\lambda=1}^{3} \left(\sum_{k=0}^{\lfloor \frac{L}{2} \rfloor} N^{L-2k} A_\lambda^{(L,2k)} \right) C_\lambda + \sum_{\lambda=4}^{6} \left(\sum_{k=0}^{\lfloor \frac{L-1}{2} \rfloor} N^{L-2k-1} A_\lambda^{(L,2k+1)} \right) C_\lambda$$

Independent components

$\lambda = 1, 2, 3$ $\rho = 4, 5, 6$

- leading power of N $A_\lambda^{(L,0)}$
- subleading powers of N $A_\rho^{(1,1)}$

$A_\lambda^{(2,2)}$

$A_\lambda^{(3,2)}$ $A_\rho^{(3,3)}$ $A_\rho^{(3,1)}$

- related by color identities [Bern, Kosower; Naculich]

e.g. U(1) decoupling identity

$$A_4^{(1,1)} = A_5^{(1,1)} = A_6^{(1,1)} = 2 \sum_{\lambda=1}^{3} A_\lambda^{(1,0)}$$

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independent terms

Infrared structure (1)

infrared divergences controlled by RG equation for Wilson lines

$$\mathcal{A}(p_i,\epsilon) = \mathbf{Z}(p_i,\epsilon)\mathcal{A}^f(p_i,\epsilon)$$

$$\mathbf{Z}(p_i,\epsilon) = \mathcal{P}exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\mu^2}{\mu^2}\mathbf{\Gamma}(p_i,\mu^2,\alpha(\mu^2))\right\}$$

in N=4 SYM integral can be done explicitly

$$\frac{1}{4}\sum_{L=1}^{\infty}\alpha^L\left[\frac{\gamma_c^{(L)}}{L^2\epsilon^2}\mathbf{D_0} - \frac{\gamma_c^{(L)}}{L\epsilon}\mathbf{D} + \frac{4}{L\epsilon}\gamma_J^{(L)}\mathbb{I} + \frac{1}{L\epsilon}\mathbf{\Delta}^{(L)}\right]$$

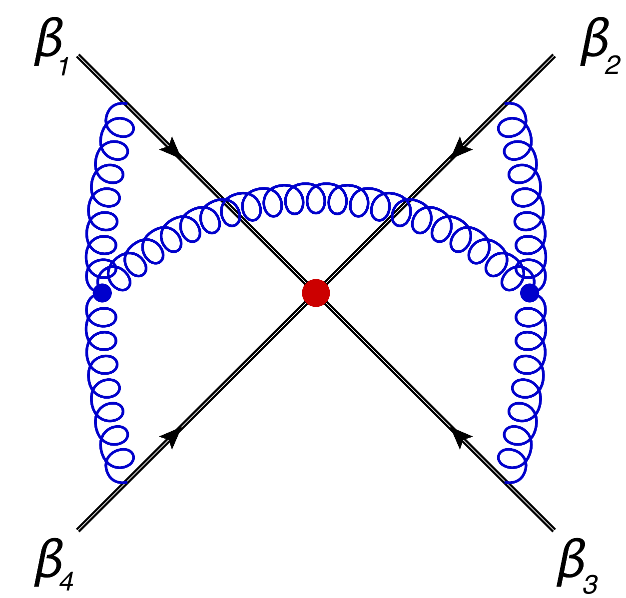
2 loops: dipole formula [Catani; Teyeda-Yeomans, Sterman]

$$\mathbf{D}_0 = \sum_{i\neq j}\mathbf{T}_i\cdot\mathbf{T}_j, \quad \mathbf{D} = \sum_{i\neq j}\mathbf{T}_i\cdot\mathbf{T}_j\log\left(\frac{-s_{ij}}{\mu^2}\right)$$

corrections at 3 loops [Almelid, Duhr, Gardi, 2015]

Infrared structure (2)

corrections at 3 loops [Almelid, Duhr, Gardi, 2015]



$$\Delta_n^{(3)}\left(\{\rho_{ijkl}\}\right) = 16 f_{abe} f_{cde}\Big\{-C\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n\\ j,k\neq i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c \; + \tag{4.1}$$

$$\sum_{1\le i<j<k<l\le n}\Big[\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\,\mathscr{F}(\rho_{ikjl},\rho_{iljk}) + \mathbf{T}_i^a\mathbf{T}_k^b\mathbf{T}_j^c\mathbf{T}_l^d\,\mathscr{F}(\rho_{ijkl},\rho_{ilkj}) + \mathbf{T}_i^a\mathbf{T}_l^b\mathbf{T}_j^c\mathbf{T}_k^d\,\mathscr{F}(\rho_{ijlk},\rho_{iklj})\Big]\Big\},$$

for n=4 points we extract

$$\mathbf{\Delta}_4^{(3)} = 4\, f_{abe} f_{cde}\Big[\mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d\,\mathcal{S}(x) + \mathbf{T}_4^a\mathbf{T}_1^b\mathbf{T}_2^c\mathbf{T}_3^d\,\mathcal{S}(1/x)\Big],$$

$$\mathbf{\Delta}_3^{(3)} = -C\, f_{abe} f_{cde}\sum_{\substack{i=1\dots4\\ 1\le j<k\le 4\\ j,k\neq i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c\,.$$

$$\begin{aligned}\mathcal{S}(x) = {} & \\ & 2H_{-3,-2} + 2H_{-2,-3} - 2H_{-3,-1,-1} + 2H_{-3,-1,0} \\ & -2H_{-2,-2,-1} + 2H_{-2,-2,0} - 2H_{-2,-1,-2} - H_{-1,-2,-2} \\ & -H_{-1,-1,-3} + 4H_{-2,-1,-1,-1} - 2H_{-2,-1,-1,0} \\ & -H_{-1,-2,-1,0} - H_{-1,-1,-2,0} + \zeta_3 H_{-1,-1} + 4\zeta_3\zeta_2 - \zeta_5 \\ & +\zeta_2(6H_{-3} - 10H_{-2,-1} + 6H_{-2,0} - H_{-1,-2} - H_{-1,-1,0}) \\ & +i\pi\Big[2H_{-3,-1} - 2H_{-3,0} + 2H_{-2,-2} - 4H_{-2,-1,-1} \\ & \quad + 2H_{-2,-1,0} - 2H_{-2,0,0} + H_{-1,-2,0} + H_{-1,-1,0,0} \\ & \quad + \zeta(3H_{-1,-1} - 4H_{-2}) - \zeta_3 H_{-1}\Big].\end{aligned} \tag{13}$$

$$x = t/s \qquad u = -s-t > 0$$

H: harmonic polylogarithms

predicts infrared divergences of any massless 4-particle amplitude

finite part

- infrared-finite part $$\mathcal{H} = \lim_{\epsilon\to 0} \mathcal{A}^f \, .$$

- planar contribution given by simple formula to all orders

$$\sum_L \alpha^L H_1^{(L,0)} = H_1^{(0,0)} \exp\Big\{ -\frac{N_c \gamma_c(\alpha)}{2} \log\frac{-s}{\mu^2} \log\frac{-t}{\mu^2} - \frac{\gamma_J(\alpha)}{2}\Big[\log\frac{-s}{\mu^2} + \log\frac{-t}{\mu^2}\Big] + C(\alpha)\Big\} , \qquad (16)$$

[conjecture: Bern, Dixon, Smirnov, 2005]

[proof: Drummond, JMH, Korchemsky, Sokatchev, 2008]

- independent non-planar contributions

2 loops: $H_1^{(2,2)}$

3 loops: $H_1^{(3,2)}$ $\quad H_4^{(3,1)}$

independent function at 2 loops

$$
\begin{aligned}
\mathcal{H}_1^{(2,2)} = \frac{i\mathcal{K}}{x}\Bigg\{ & 18\zeta_2 H_{-1,0} + 24\zeta_2 H_{0,0} - 8H_{-3,-1} + 6H_{-3,0} - 6H_{-2,-2} + 2H_{-1,-3} - 2H_{-2,-1,-1} \\
& -6H_{-2,-1,0} + 2H_{-2,0,0} - 6H_{-1,-2,-1} + 2H_{-1,-2,0} - 10H_{-1,-1,-2} + 8H_{-1,-1,-1,-1} \\
& -10H_{-1,-1,-1,0} + 4H_{-1,-1,0,0} - 2H_{-1,0,0,0} - 6\zeta_2 H_{-2} - 2\zeta_3 H_{-1} + 6H_{-4} \\
& +i\pi\Bigg[2H_{-2,-1} + 6H_{-2,0} + 6H_{-1,-2} - 8H_{-1,-1,-1} + 10H_{-1,-1,0} - 2H_{-1,0,0} \\
& \quad -6H_{0,0,0} - 14H_{-1}\zeta_2 + 8H_{-3} - 6\zeta_3 \Bigg]\Bigg\} \\
\frac{i\mathcal{K}}{1+x}\Bigg\{ & -36\zeta_2 H_{-1,0} - 12\zeta_2 H_{0,0} + 8H_{-3,-1} - 8H_{-3,0} + 4H_{-2,-2} - 4H_{-2,-1,-1} \\
& +4H_{-2,-1,0} + 4H_{-1,-2,-1} + 12H_{-1,-1,-2} + 12H_{-1,-1,-1,0} - 4H_{-1,-1,0,0} \\
& -4H_{-1,0,0,0} + 4H_{0,0,0,0} - 78\zeta_4 + 12\zeta_2 H_{-2} + 4\zeta_3 H_{-1} - 8H_{-4} \\
& +i\pi\Bigg[4H_{-2,-1} - 4H_{-2,0} - 4H_{-1,-2} - 12H_{-1,-1,0} + 8H_{0,0,0} - 4\zeta_2 H_{-1} \\
& \quad +16\zeta_2 H_0 - 8H_{-3} \Bigg]\Bigg\}
\end{aligned}
$$

[appeared in different form in Naculich, Nastase, Schnitzer, 2013]

uniform weight 4; similarly, two new weight 6 functions at 3 loops

Regge limit

take $s \gg t$

useful to decompose into irreducible representations in t-channel

$\mu^2 = -t$

in octet channel we find

$$\mathcal{A}_{\mathbf{8}_a} \sim s^{w_{\mathbf{8}_a}}$$

$$\begin{aligned} w_{\mathbf{8}_a}|_{\alpha^3} =& N_c^3 \left[\frac{11\zeta_4}{48}\frac{1}{\epsilon} + \frac{5}{24}\zeta_2\zeta_3 + \frac{1}{4}\zeta_5 + \mathcal{O}(\epsilon) \right] \\ &+ N_c \left[\frac{\zeta_2}{4}\frac{1}{\epsilon^3} - \frac{15\zeta_4}{16}\frac{1}{\epsilon} - \frac{77}{4}\zeta_2\zeta_3 + \mathcal{O}(\epsilon) \right] \end{aligned}$$

also predictions for other channels

can be used to test Regge theory and determine parameters, in particular 3-loop Regge trajectory

Regge limit

take $s \gg t$

Regge limit of finite part

$$\mathcal{H} = \sum_{k,q} \alpha^k \left(\log \frac{s}{t}\right)^q \mathbf{O}_{k,q} \mathcal{H}^{(0)} + \mathcal{O}(1)$$

$$\mathbf{O}_{2,1} = -\frac{1}{8}\zeta_3 \mathbf{T}^2 \,, \tag{19}$$

$$\mathbf{O}_{3,2} = i\pi \frac{11}{24}\zeta_3 [[\mathbf{S},\mathbf{T}],\mathbf{T}] \,, \tag{20}$$

$$\begin{aligned}\mathbf{O}_{3,1} =& i\pi \frac{1}{16}\zeta_4 \left(3[\mathbf{S},\mathbf{T}]\mathbf{T} + 58[[\mathbf{S},\mathbf{T}],\mathbf{T}]\right) \\ &+ \frac{11}{6}\zeta_2\zeta_3 \left(3[\mathbf{S},\mathbf{T}]\mathbf{T} + 2[[\mathbf{S},\mathbf{T}],\mathbf{T}] - [\mathbf{S}^2,\mathbf{T}]\right) \\ &+ \left(\frac{1}{4}\zeta_5 - \frac{1}{24}\zeta_2\zeta_3\right)\mathbf{T}^3 - 4\zeta_2\zeta_3\mathbf{T} \,. \end{aligned} \tag{21}$$

color operators $\mathbf{S} = (\mathbf{T}_1 + \mathbf{T}_2)^2 \qquad \mathbf{T} = (\mathbf{T}_2 + \mathbf{T}_3)^2$

Conclusion

- milestone in perturbative QFT:
  full 3-loop four-gluon scattering amplitude

- independent verification of 3-loop soft anomalous dimension matrix
  predicts infrared divergences of any massless four-particle amplitude

- non-trivial data point for the study of non-planar scattering amplitudes