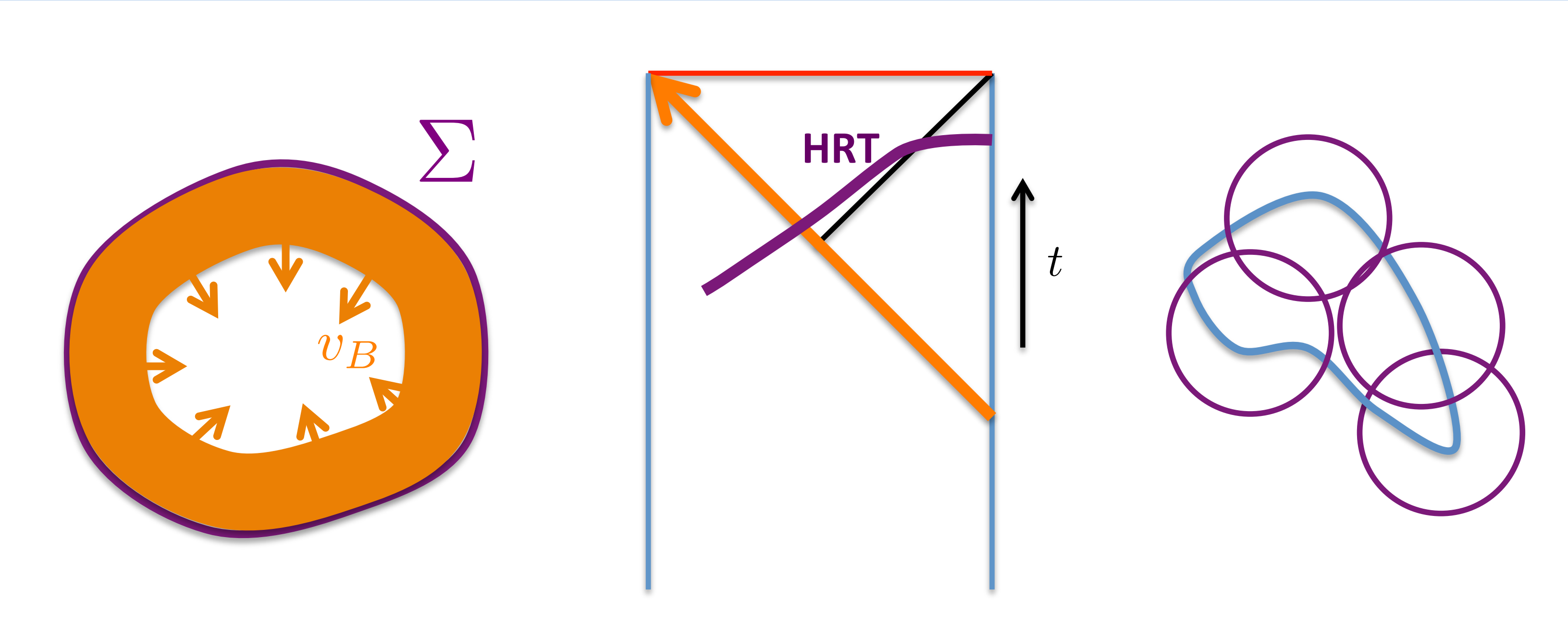
Spread of entanglement and chaos



Márk Mezei (Princeton)

MM, Stanford [to appear]; MM [to appear];
Casini, Liu, MM [1509.05044]; Cotler, Hertzberg, MM, Mueller [to appear]

Strings 2016

Outline

Entanglement generation and chaos
- Two velocities
- Bounds

Data on entanglement growth
- Holographic results
- Spin chain results

Interpretation and benchmarking
- Operator growth model
- Free streaming, free scalar theory

Summary and open questions

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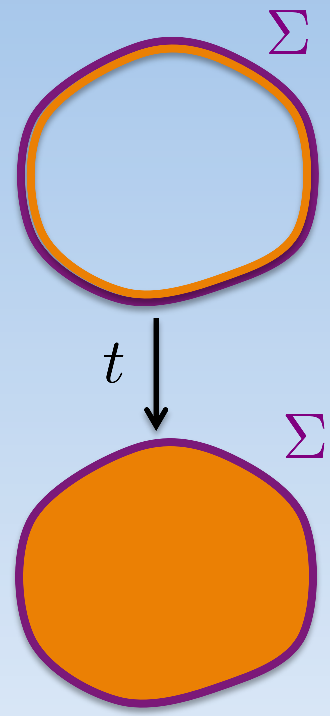
Entanglement generation in global quenches

Global quench:

- Thermalization in a pure state $|\psi(t)\rangle$
- Start with QFT in a short-range entangled state at t=0. (E.g. inject uniform energy density or change the Hamiltonian)
- One-point functions reach thermal value $t_{\rm loc} \sim 1/T$
- EE (similarly to $\langle \phi(R)\, \phi(0) \rangle$) take $t_s \sim R$ to saturate to thermal value
- Good diagnostic of thermalization is how close $\rho_A\left(|\psi(t)\rangle\right)$ is to $\mathrm{Tr}_{\bar{A}} e^{-\beta(E)\, H}$



$$S_0 = \frac{A_\Sigma}{\delta^{d-2}} + \ldots$$

Typical point inside is unentangled with outside

$$S_{\rm eq} = s_{\rm th}\, V_A + \ldots$$

Typical point inside is entangled with outside

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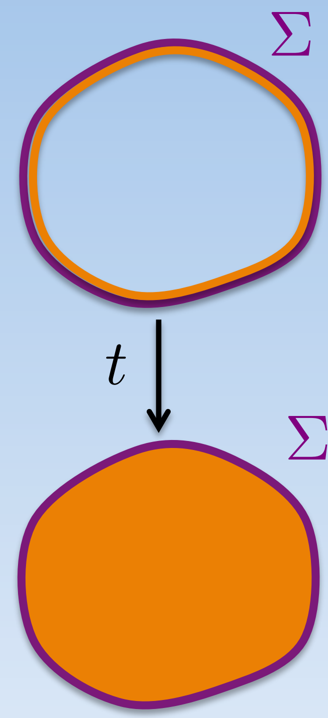

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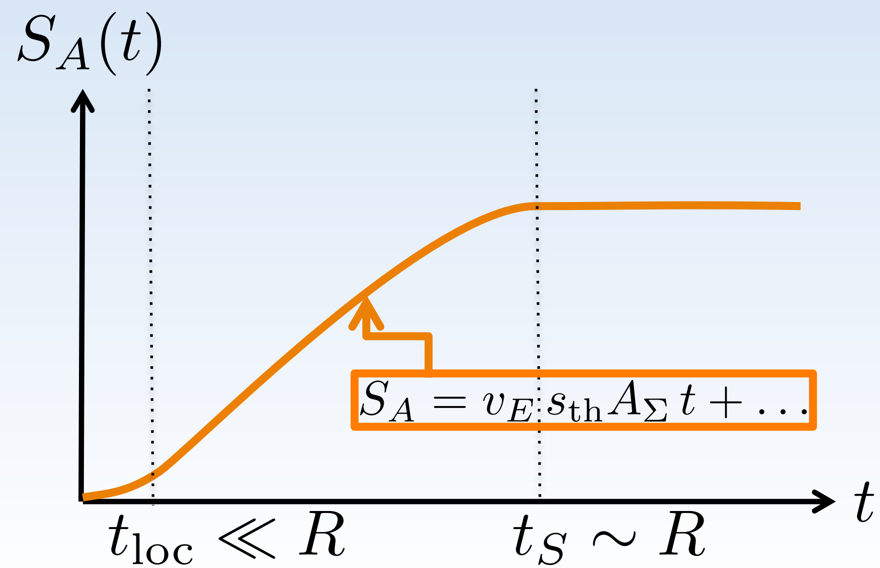
Typical point inside is entangled with outside

What is the time evolution of EE?

- 2d: numerics, CFT techniques [Huse, Kim; MM, Stanford; Calabrese, Cardy]
- d>2: holography, free field theory [Hartman, Maldacena; Liu, Suh; Cotler, Hertzberg, MM, Mueller]

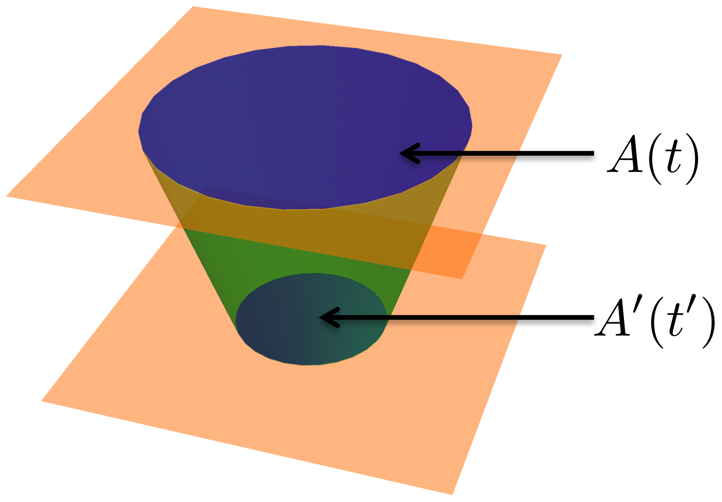


Bounds on entanglement growth

Monotonicity of relative entropy combined with emergent light cones

- v_B cone at finite temperature in chaotic systems
- Monotonicity of relative entropy for subsystems
- Tsunami bound [Afkhami-Jeddi, Hartman]

$$S[A(t)] \leq S[A'(t')] + s_{\mathrm{th}} \left(V[A(t)] - V[A'(t')] \right)$$



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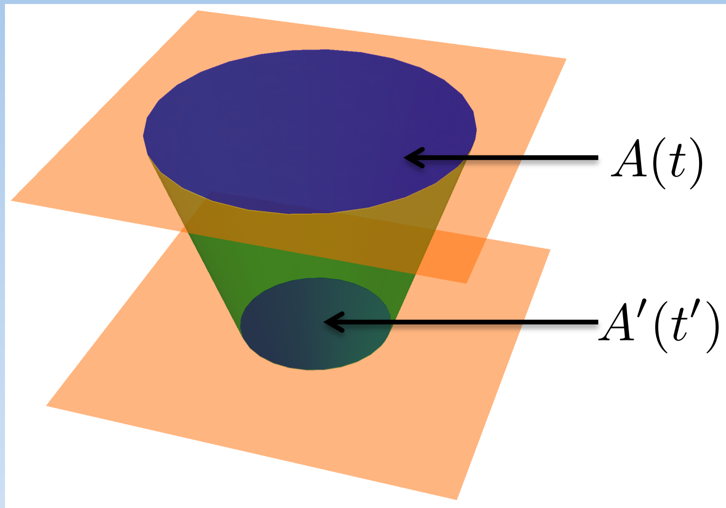

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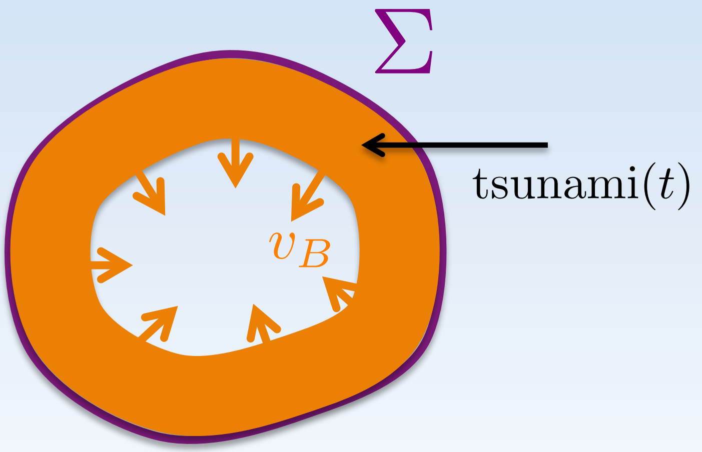

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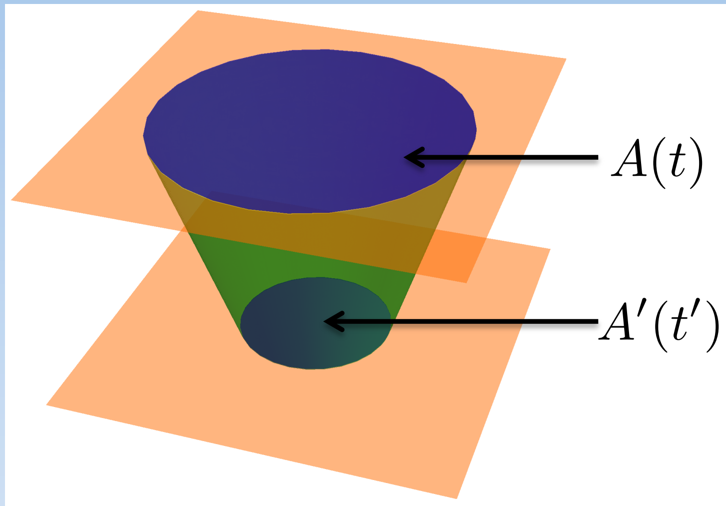

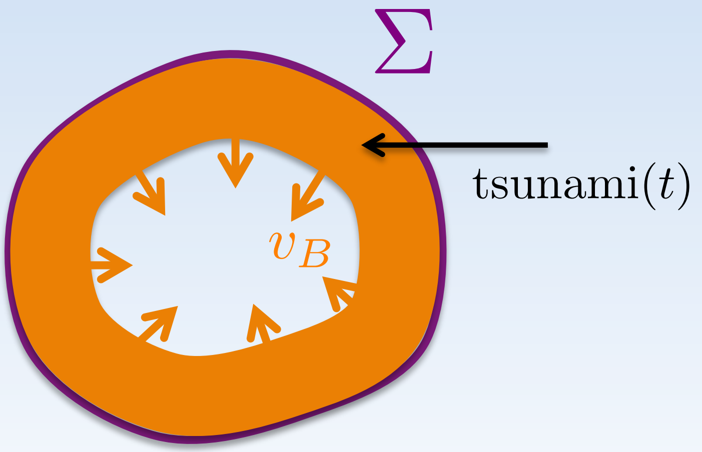

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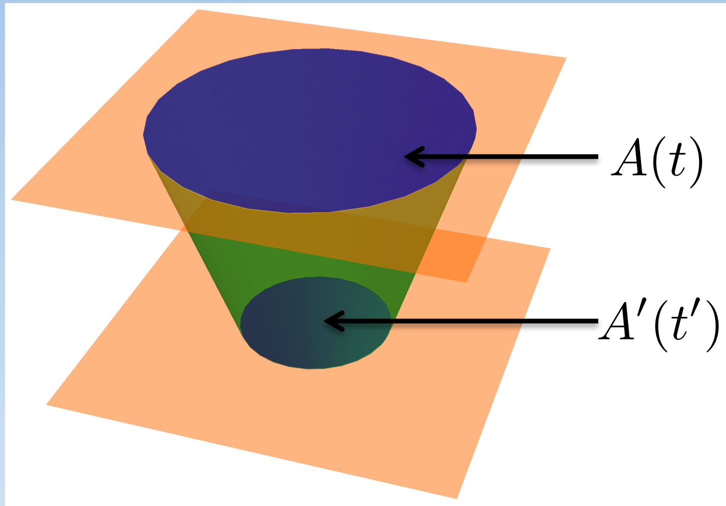
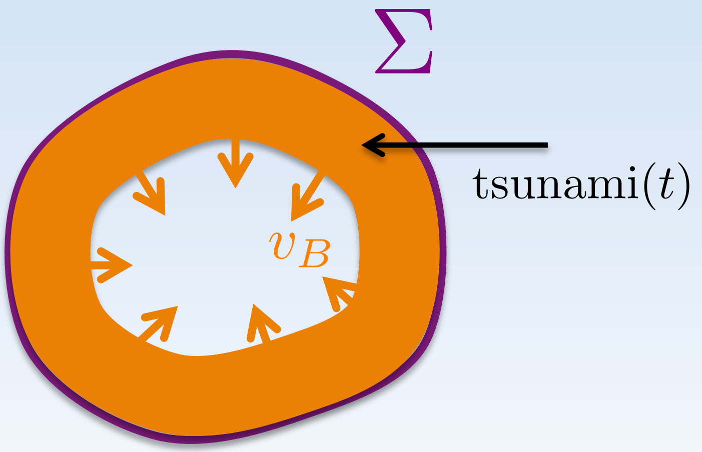


Proposed inequality

$$\partial_t S[A(t)] \leq v_E\, s_{\rm th}\, A_\Sigma$$

- Rigorous versions exist for lattice systems
- Can be proven in holography [MM]



Combination of the two bounds captures many of the essential details of entanglement growth in chaotic systems.

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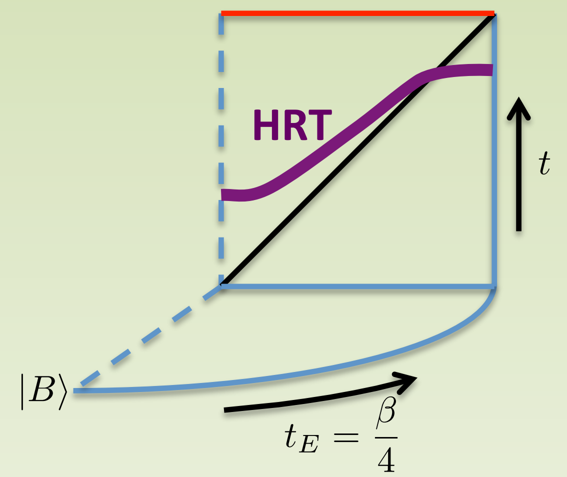
Holographic results on entanglement

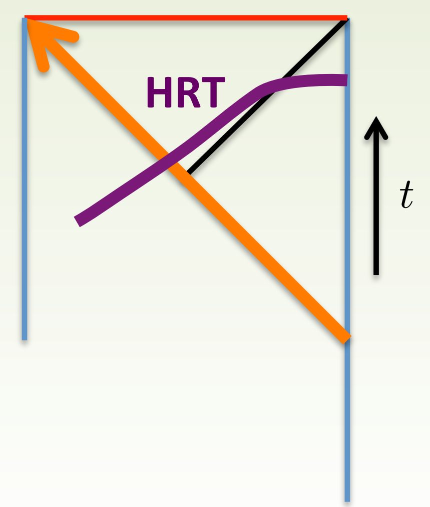
Holographic models of quenches

- Dual of Cardy-Calabrese boundary state is eternal BH with end of world brane [Hartman, Maldacena]
- Injecting energy density is dual to a collapsing shell. Saturation happens when the HRT surface touches the shell [Liu, Suh]

End of the world brane quench



Vaidya quench



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- v_E is determined by behind the horizon physics
- Saturation is determined by near horizon physics, and EE saturates as fast as possible

$$t_S = \frac{R}{v_B}$$

Conceptual argument based on entanglement wedge reconstruction.

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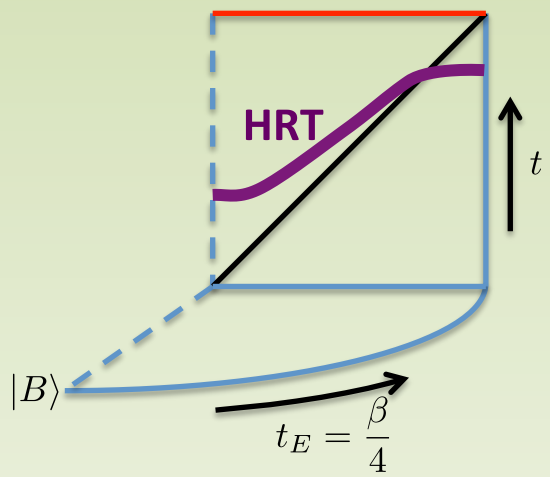

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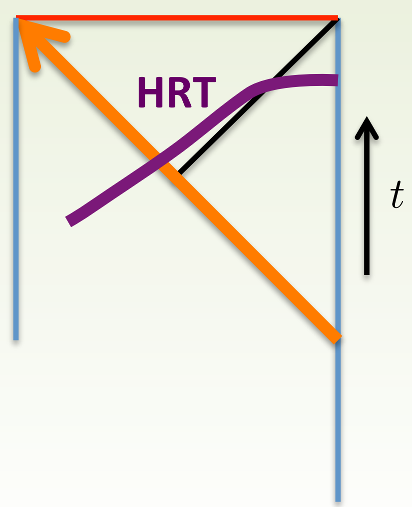

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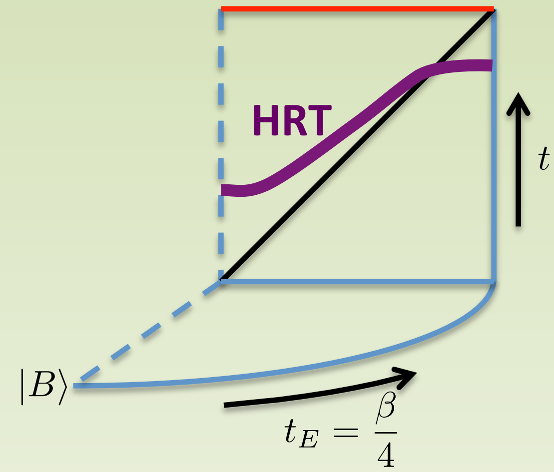
Conceptual argument based on entanglement wedge reconstruction.

- Using the NEC, we can show that there are non-trivial constraints on these velocities:

$$v_E \leq v_E^{(\mathrm{SBH})}, \quad v_B \leq v_B^{(\mathrm{SBH})}, \quad v_E \leq v_B$$

End of the world brane quench



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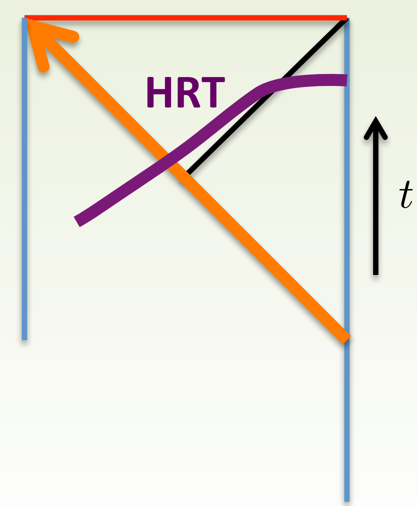
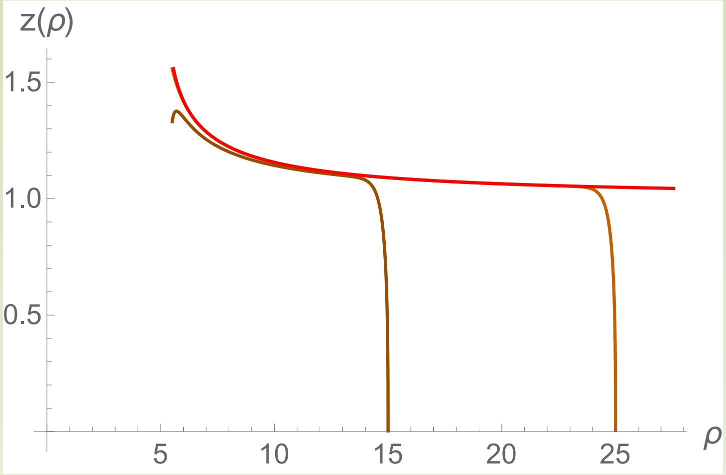


Holographic results on entanglement

Detailed understanding of how HRT surfaces are behaving

- For large R, we can understand the entropy analytically
- In both setups the minimal surfaces are close to a critical surface determined by an **algebraic equation**.
- They shoot out to the boundary exponentially fast.

Critical surface



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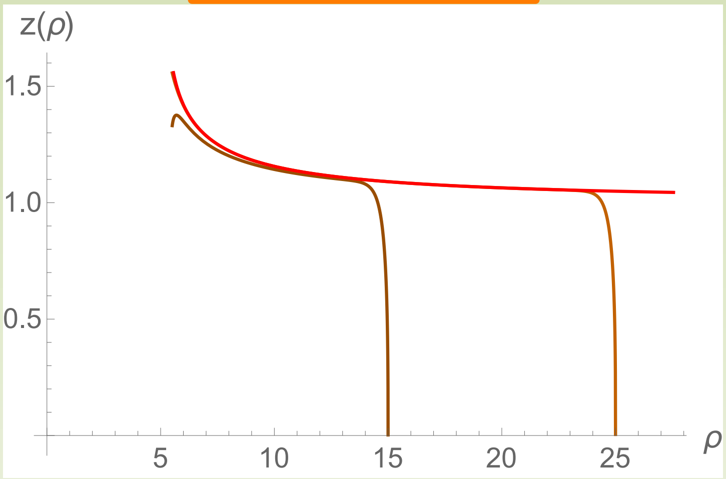
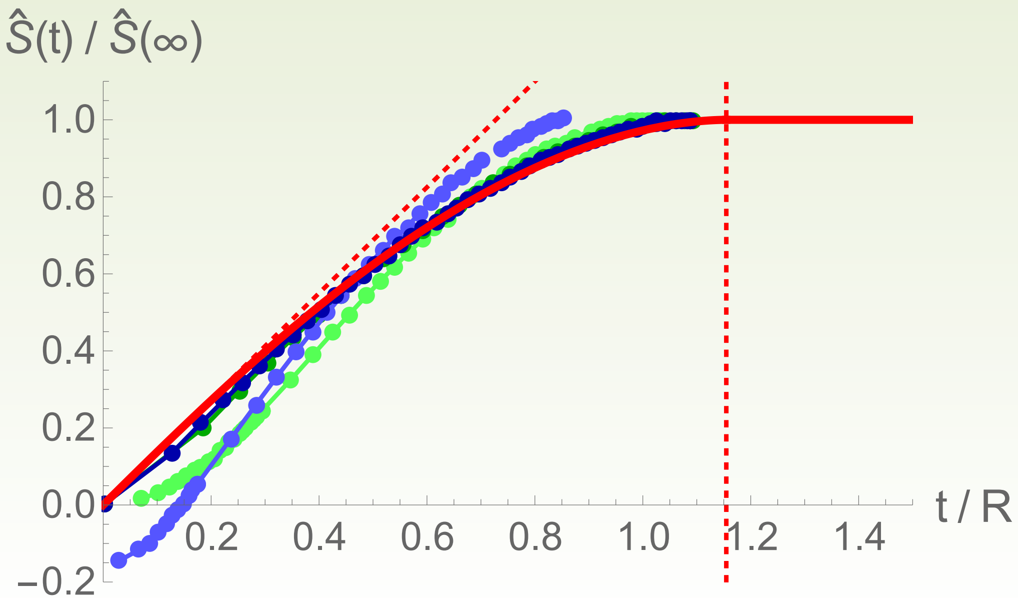


Detailed understanding of how HRT surfaces are behaving

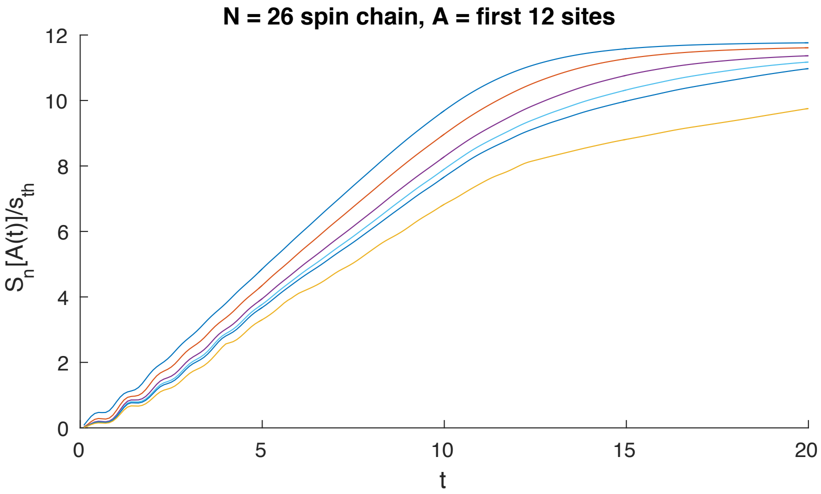
- For large R, we can understand the entropy analytically
- In both setups the minimal surfaces are close to a critical surface determined by an **algebraic equation**.
- They shoot out to the boundary exponentially fast.
- Entropy and time are given by the critical surface

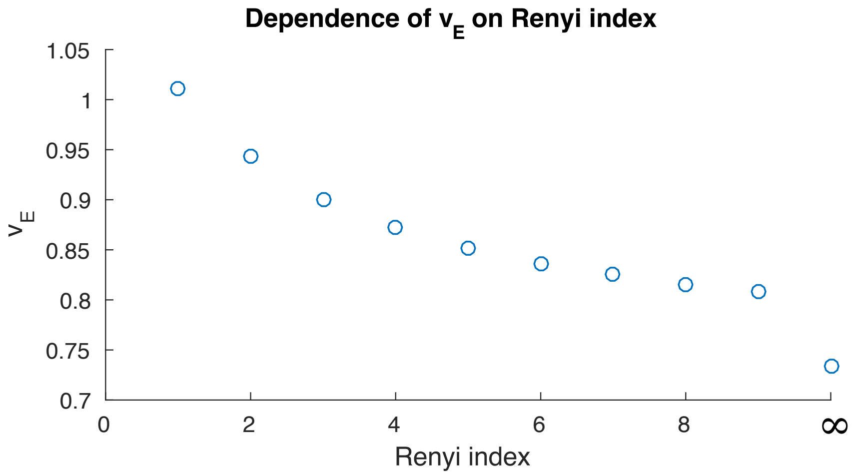


Spin chain results on entanglement and chaos

Chaotic spin chain Hamiltonian: $H = -\sum_i (Z_i Z_{i+1} - 1.05 X_i + 0.5 Z_i)$

- Entropy growth and v_E:





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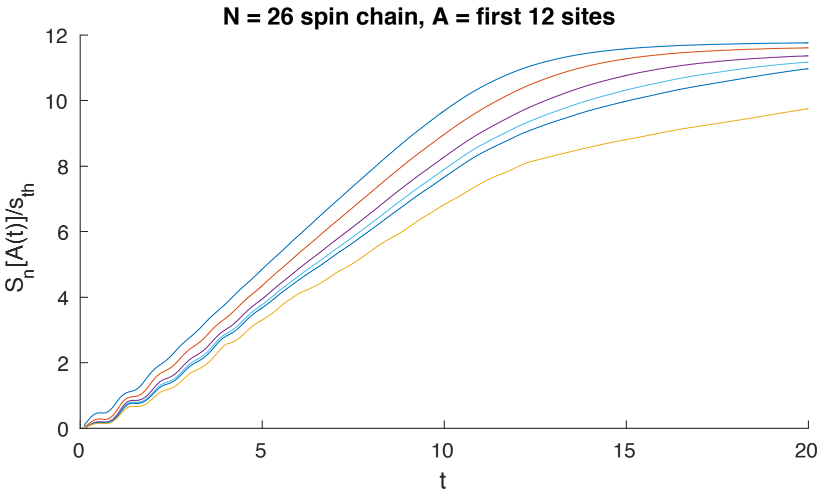
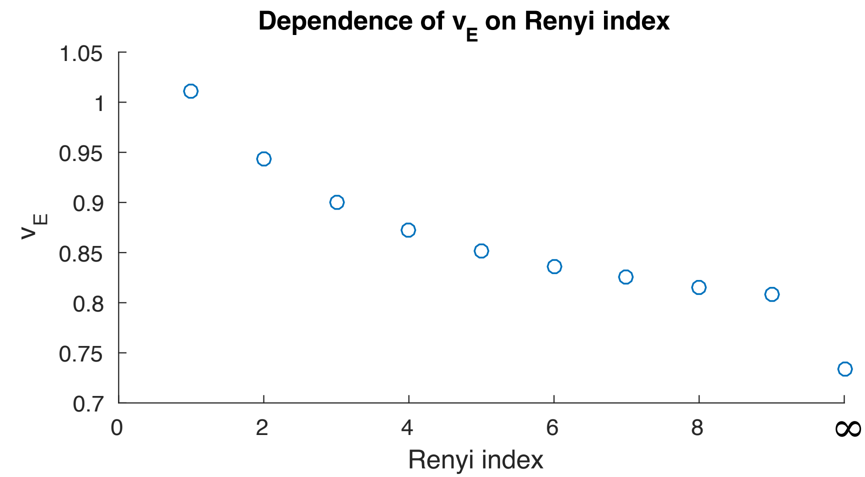
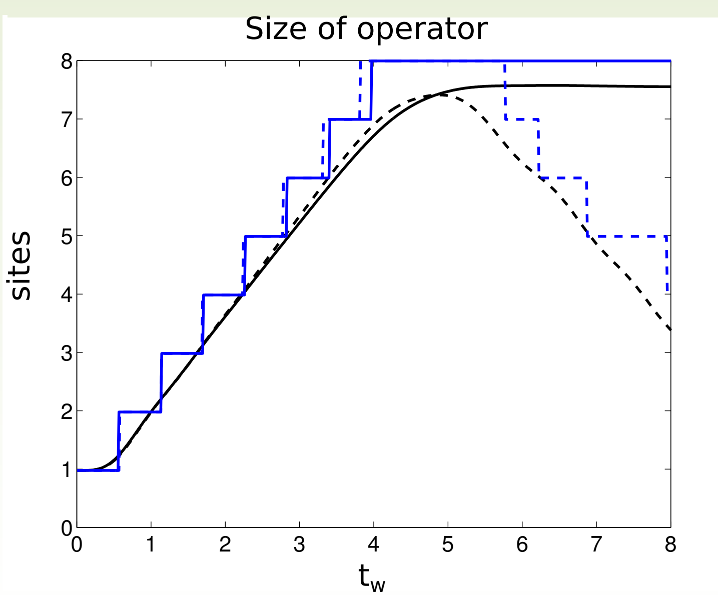




- Operator growth [Roberts, Susskind, Stanford]

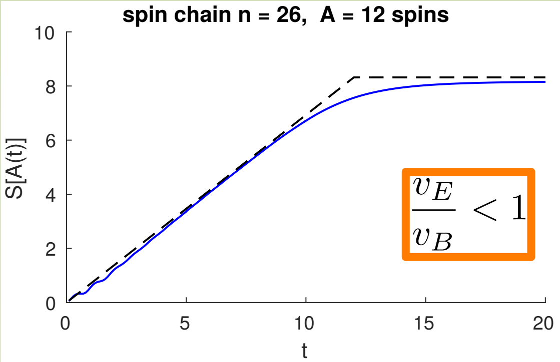


$$v_B = 2.0 > v_E\,, \quad t_S > \frac{R}{v_B}$$

Comparison with bounds

Combination of the two bounds comes very close to the data from chaotic systems.

- d=2: linear growth until saturation



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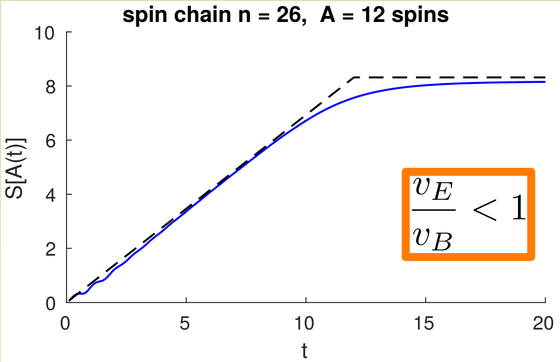
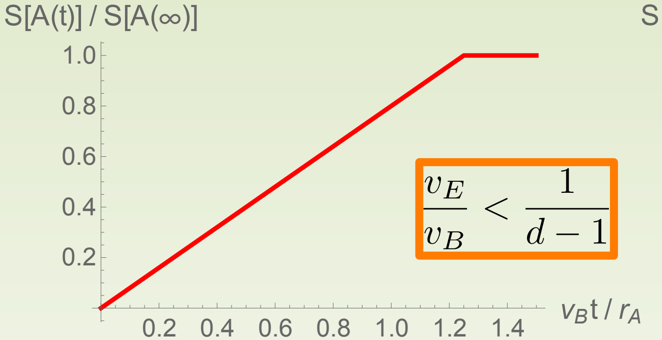
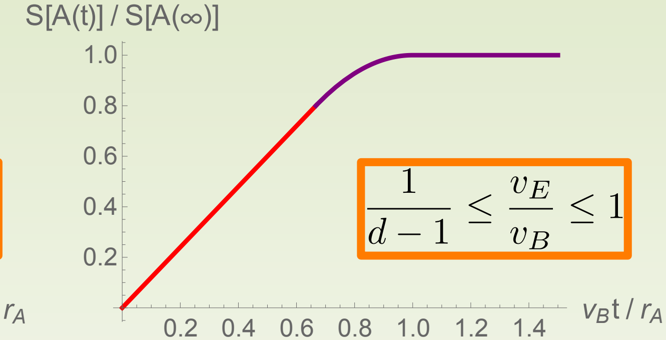
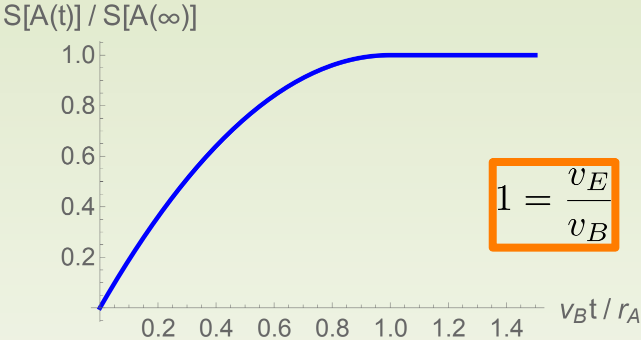


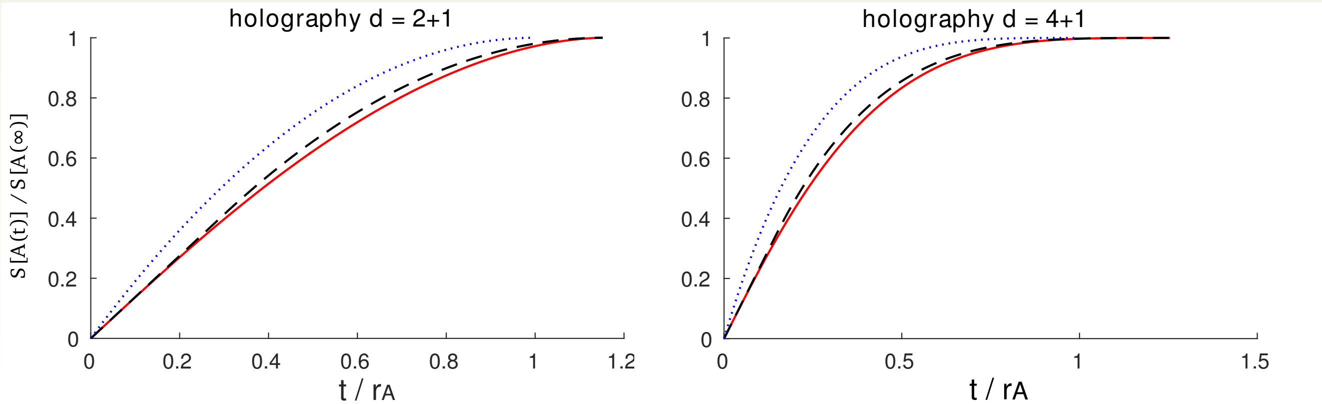
- d>2: three regimes







Middle regime in good agreement with holographic theories



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Operator growth model

Operator counting model [Abanin, Ho]

- Closer in spirit to spin chains, infinite temperature
- The reduced density matrix is an operator, so it also spreads

$$\rho(0) = |\uparrow\uparrow \dots \uparrow\rangle\langle\uparrow\uparrow \dots \uparrow| = \prod_i \frac{\mathbb{I}_i + Z_i}{2} = \frac{1}{2^{V/2}} \sum_{\mathcal{O}(0)} \mathcal{O}(0)$$

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- Second Rényi entropy:

$$\mathrm{Tr}_A \rho_A(t)^2 \approx \frac{1}{2^{V_A}} \sum_{\mathcal{O}(0)} \mathrm{Tr}_A \left(\mathcal{O}(t)_A^2\right)$$

- Small operators contribution: 1
  Big operators: probability of staying inside $\mathrm{Tr}_A \left(\mathcal{O}(t)_A^2\right) = 2^{-\alpha\, s_{\mathrm{th}} A[\mathcal{O}(0)](t-t_{\mathrm{delay}})}$
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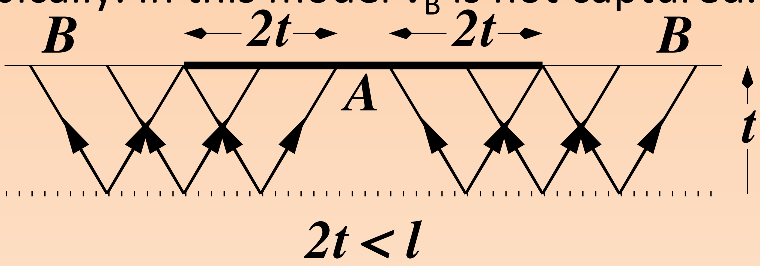
- Saturates the combined bounds, gives microscopic picture for them
  - t_S is determined by when the last small operator gets out
  - In the spin chain we can measure α independently, good agreement with the data for $S_2(t)$

Free streaming model of entanglement spread

Calabrese-Cardy model: energy injection from quench creates a finite density of EPR pairs, subsequently travel freely at the speed of light isotropically. In this model v_B is not captured.

- Leads to linear growth with $v_E = 1$ in 2d



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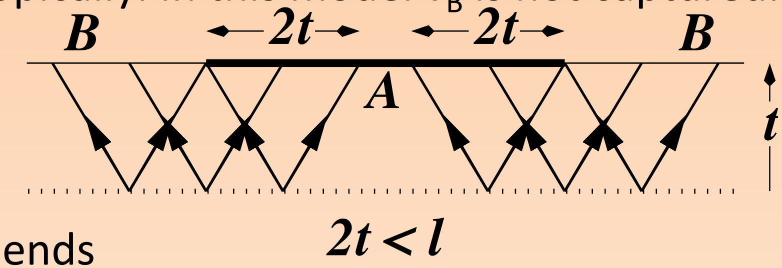
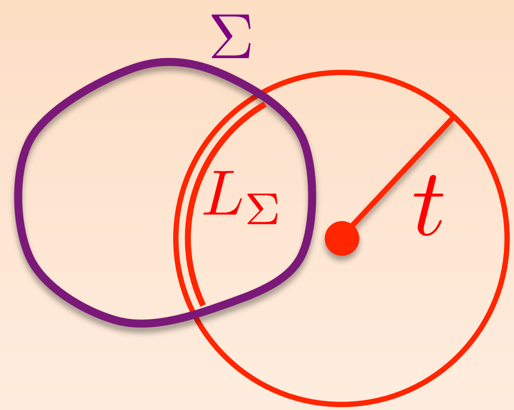


- Higher dimensions: entanglement spreading depends on entanglement pattern on the light cone $\mu[L_\Sigma]$
Contribution from each light cone has to be added.



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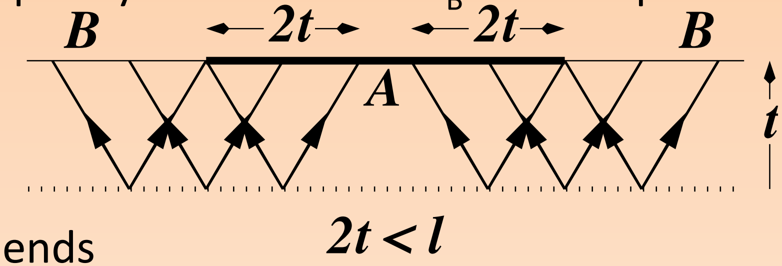

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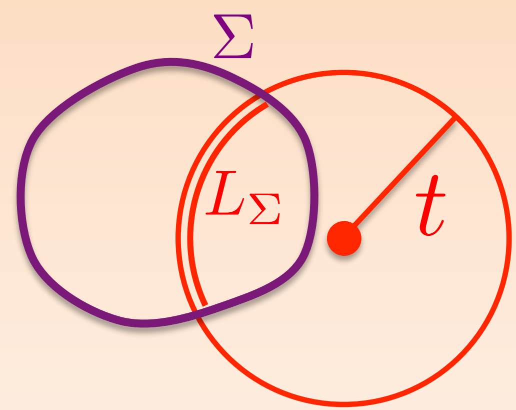

Bound on the entanglement velocity from SSA:

$$v_E \leq v_E^{(\mathrm{EPR})} = \frac{\Gamma(\frac{d-1}{2})}{\sqrt{\pi}\Gamma(\frac{d}{2})} < v_E^{(\mathrm{SBH})}$$

Slower than holography.

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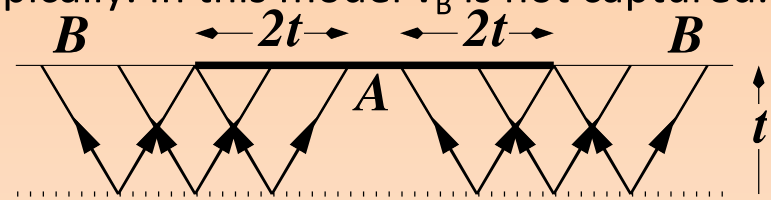

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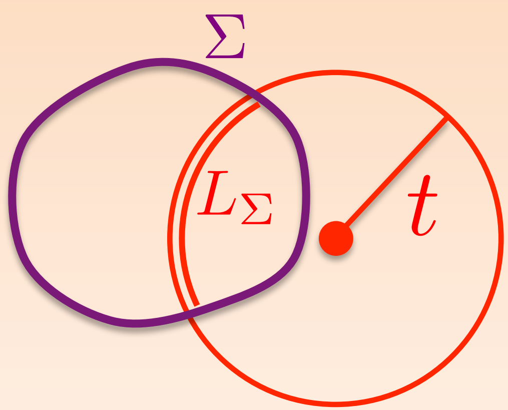

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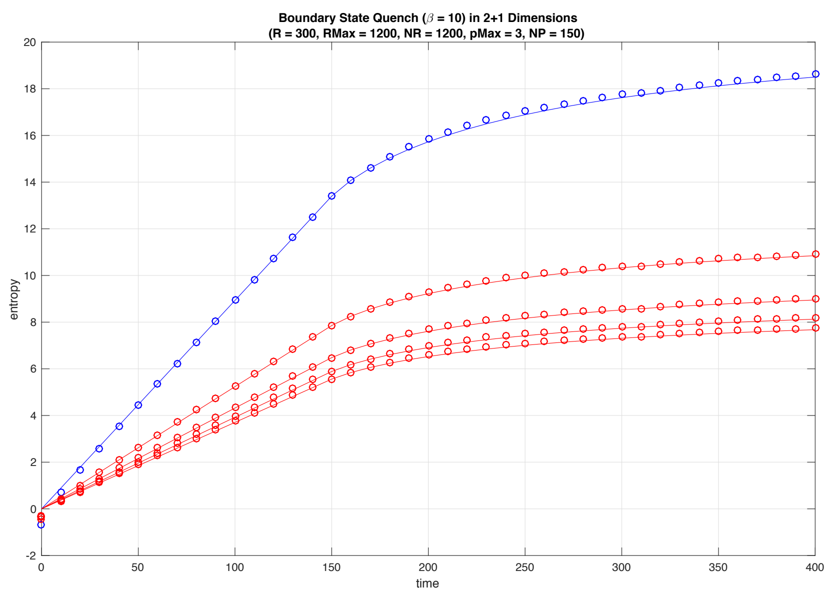
- In strongly coupled systems, entanglement propagates faster than what's possible for free particles streaming at the speed of light!
- $t_S^{(\mathrm{SBH})} > t_S$ is achievable, makes free streaming look even less effective
- Consider the effect of interactions: tensor network picture emerging from scattering particles is natural [Hartman, Maldacena; Casini, Liu, MM]

Entanglement spread in free scalar theory

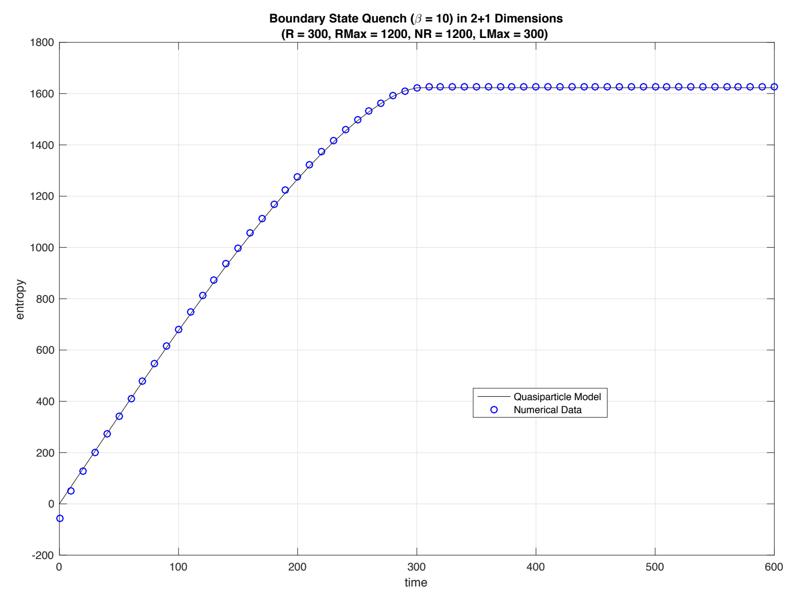
In a free theory for time dependent Gaussian states the symplectic eigenvalues of the (reduced) correlation matrix determine the entanglement entropy.

- Numerical results for 3d boundary state quench for scalar field [Cotler, Hertzberg, MM, Mueller]

Strip



Sphere



Outline

Entanglement generation and chaos
- Two velocities
- Bounds

Data on entanglement growth
- Holographic results
- Spin chain results

Interpretation and benchmarking
- Operator growth model
- Free streaming, free scalar theory

Summary and open questions

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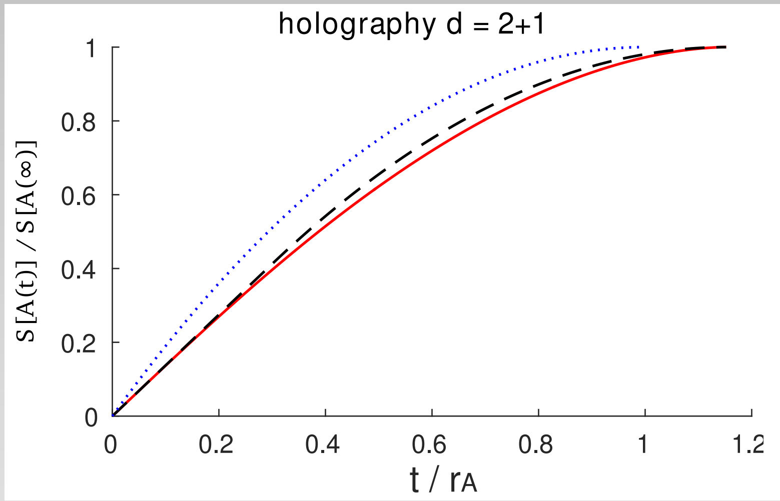
- Studied EE spread in a global quench
- Bound from chaos and thermal relative entropy:
  $v_E \leq v_B\,, \quad t_S \geq R/v_B$
- In holography: $\boxed{t_S = R/v_B}$
- Can solve for the entire $S_A(t)$ curve analytically
- In chaotic spin chain: $v_E < v_B$
- Operator growth model saturates the bounds, good agreement with holography and the spin chain
- Free streaming is slower than holography, quasiparticle picture agrees with free theory



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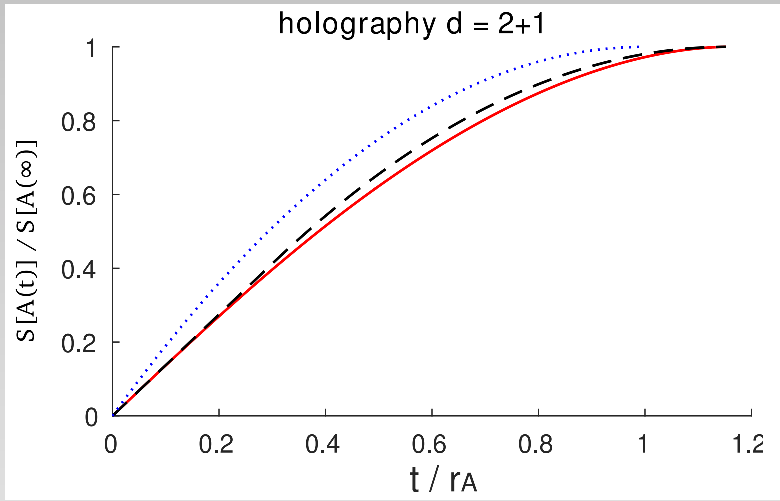

Open questions

- **What is an independent characterization of v_E?**
- Can the bound from relative entropy be saturated in a QFT? Are the holographic bounds $v_E \leq v_E^{(\mathrm{SBH})}\,, \quad v_B \leq v_B^{(\mathrm{SBH})}$ universal?
- The velocities and t_s are new observables in a QFT. Are they calculable?
  - What are they in weakly coupled theories? [v_B: Stanford]
  - What are they for perturbed 2d CFTs? [v_E: Cardy]