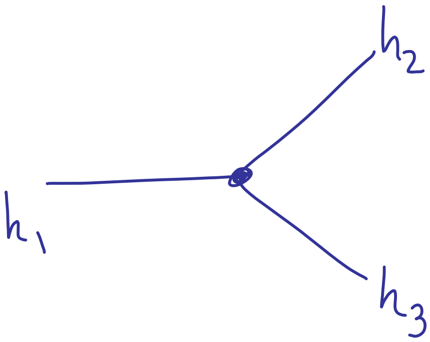
# Towards Deriving String Theory as the Weakly Coupled UV Completion of Gravity

w/ Yutin Huang
Tzu-Chen Huang

# Massless Particle Dynamics: Fixed by Poincaré + Unitarity




$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$$

"Chinese Magic Variables"

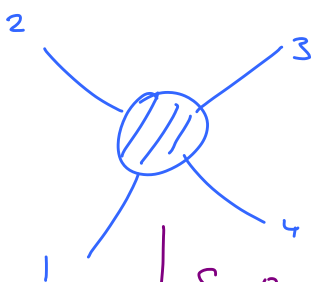
Little group: $\lambda \to t\lambda$, $\tilde{\lambda} \to t^{-1}\tilde{\lambda}$

$\lambda_1 \propto \lambda_2 \propto \lambda_3$, $\quad [12]^{h_1+h_2-h_3}\,[23]^{h_2+h_3-h_1}\,[31]^{h_3+h_1-h_2}$ $\quad h_1+h_2+h_3>0$

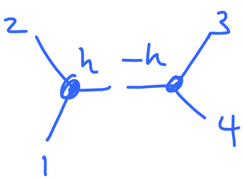
$\tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3$ $\quad \langle 12\rangle^{h_3-h_1-h_2}\,\langle 23\rangle^{h_1-h_2-h_3}\,\langle 31\rangle^{h_2-h_3-h_1}$ $\quad h_1+h_2+h_3<0$

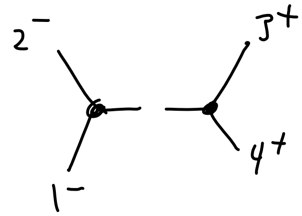
# Unitarity @ Weak Coupling

Amplitude has only poles [~ "tree level"]; Unitarity = Factorization; 4-particle test




$\downarrow s_{12}\to 0$

$\frac{1}{s_{12}}\sum_h$ 

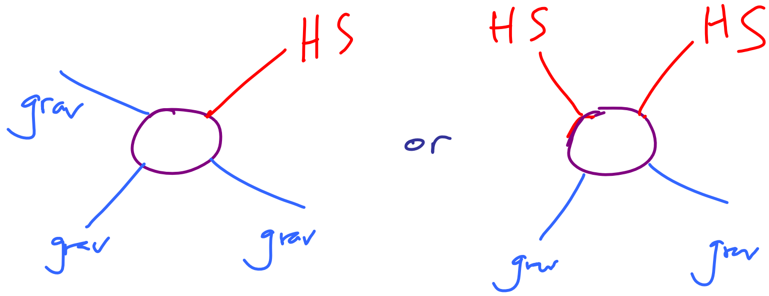

$$= \begin{cases} \dfrac{\langle 12\rangle^2[34]^2}{t} & s=1 \\ -\dfrac{\langle 12\rangle^4[34]^4}{tu} & s=2 \end{cases}$$

$$A^{s=1}[1^-2^-3^+4^+] = \frac{\langle 12\rangle^2[34]^2}{st}$$

$$A^{s=2}[1^-2^-3^+4^+] = -\frac{\langle 12\rangle^4[34]^4}{stu}$$

ALREADY "STRINGY"! [Can't separate channels]

# (Massless) Higher Spin Ruled out by GR




$\longrightarrow$ Channel residues have multiple poles in $s, t, u$; impossible to find consistent 4-pt

Only consistent 4pt: Spins $\left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$

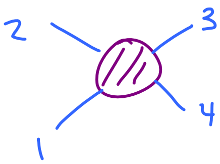
YM

$\mathcal{N} \leq 8$ SUSY

GR

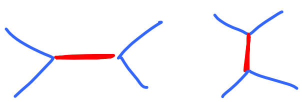
# Weakly Coupled UV Improvement/Completion

e.g. EWK "pions"




$= -\frac{1}{f^2}(s+t)$

$\rightarrow \lambda\left(\frac{s}{s-M_H^2} + \frac{t}{t-M_H^2}\right)$



Higgs

+ Also Higgs scattering ok

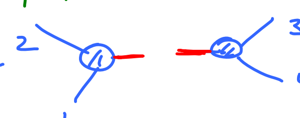
## Rules

* Only poles for $s,t,u > 0$
* Causality: fixed $t<0$, large $s$, $< s^{+2}$

[rules out dumb form factors e.g. $e^{-(s^2+t^2+u^2)}$]

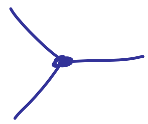
* UNITARITY/(POSITIVITY)

$\xrightarrow{s \to M^2} \frac{1}{s-M^2}$




* (Consistency for all states, massless + massive)

# Long-Range Forces $\Longrightarrow$ Challenging UV Completion

If we have  $\Longrightarrow$ massless $s, t, u$ poles

positivity $\longrightarrow$ Impossible to complete w/ finite # of massive particles.

e.g. $\phi^3$ $\quad A = g^2\left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right)$

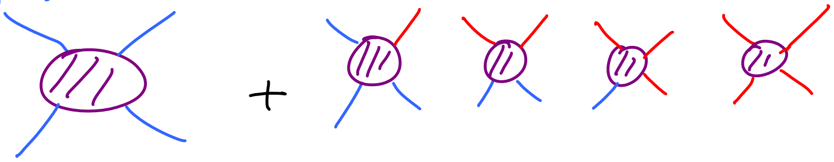
$$\to g^2\left(\frac{1}{s}\,\frac{M^2}{s-M^2} + t, u\right)$$

Negative residue!

UV completion needs $\infty$ # of particles, $\infty$ tower of spins!

# UV Completion S-Matrix Program

grav, gluons

Find  satisfying the rules

## What's new compared to 60's?

Then: massive scalars

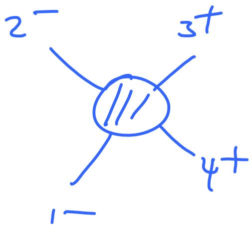
(Now we know: continuous infinity of large-N YM theories will satisfy rules)

Now: massless gravitons/gluons

* For gravity, pert. string theory is only known answer!

[* + Recall massless higher spins ruled out by gravity...]

# Simplest Guess




$$= \frac{\langle 12\rangle^4 [34]^4}{stu} \longrightarrow -\langle 12\rangle^4 [34]^4 \, F(s)F(t)F(u)$$

Kill double poles as $s \to r_i > 0$

$$\frac{\prod_I (s+z_I)(t+z_I)(-s-t+z_I)}{\prod_i (s-r_i)(t-r_i)(-s-t-r_i)}$$

$\Longrightarrow \{z_I\} \subset \{r_i + r_j\}$

Correct massless residue

$\Longrightarrow \{r_i\} = \{r_i + r_j\} = M^2 \times \{0,1,2,\ldots\}$

$$\Longrightarrow A = -\frac{\langle 12\rangle^4 [34]^4}{stu} \prod_{i=1}^{\infty} \frac{(s+i)}{(s-i)} \frac{(t+i)}{(t-i)} \frac{(u+i)}{(u-i)} = \langle 12\rangle^4 [34]^4 \, \frac{\Gamma(-s)}{\Gamma(1+s)} \frac{\Gamma(-t)}{\Gamma(1+t)} \frac{\Gamma(-u)}{\Gamma(1+u)}$$

**Matches The String Amplitude!**

# "The" weakly coupled String Amplitudes

* 4 pt tree indep of compactification

Grav: $\langle 12\rangle^4 [34]^4 \dfrac{\Gamma(1-s)}{\Gamma(1+s)}\dfrac{\Gamma(1-t)}{\Gamma(1+t)}\dfrac{\Gamma(1-u)}{\Gamma(1+u)} \times$

Type II: $-\dfrac{1}{stu}$

Het: $-\dfrac{1}{stu} + \dfrac{1}{s(1+s)}$

Bosonic: $-\dfrac{1}{stu} + \dfrac{8}{s(1+s)} - \dfrac{tu}{s(1+s)^2}$

YM: $\langle 12\rangle^2 [34]^2 \dfrac{\Gamma(1-s)\Gamma(1-t)}{\Gamma(1-s-t)} \times$

Type I: $\dfrac{1}{st}$

Bosonic: $\dfrac{1}{st} - \dfrac{u}{s(1+s)}$

[Het has massive u-channel poles]

$$\frac{\Gamma(-s)\,\Gamma(-t)\,\Gamma(-u)}{\Gamma(1+s)\,\Gamma(1+t)\,\Gamma(1+u)}$$

$$\frac{\Gamma(-s)\,\Gamma(-t)}{\Gamma(1-s-t)}$$

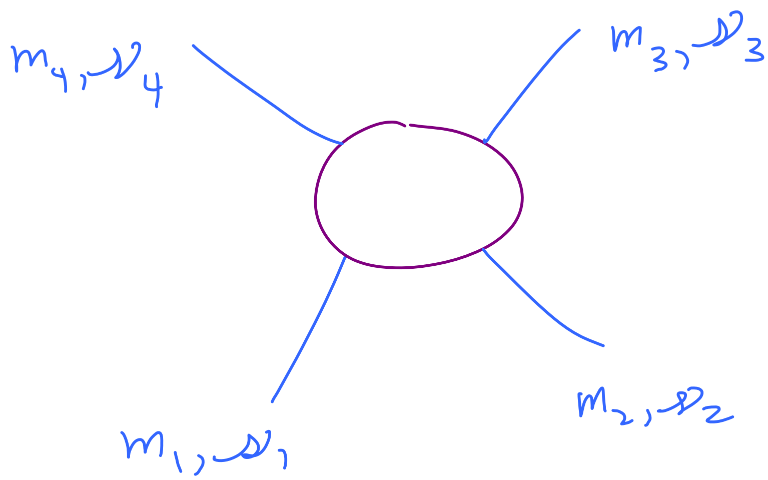
# Outline

(I) Rules for 4pt S-matrix, any masses + spins

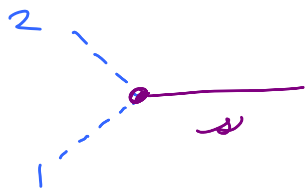
(II) First Look at Stringy magic with only massless external states + playing with deformations

(III) Open Problems + Outlook

# 4 particle S-Matrix Rules



# Easy for external scalars




$$g\,\epsilon^{\mu_1\cdots\mu_s}(p_1-p_2)_{\mu_1}\cdots(p_1-p_2)_{\mu_s}$$

symm; traceless, $p_\mu\,\epsilon^{\mu\cdots}=0$


$$\longrightarrow \frac{g\,g'}{s-M^2}\,G_s^{(d)}[\cos\theta] \quad \left\{t=\left(\frac{M^2}{2}-2m^2\right)(\cos\theta-1)\right\}$$

Gegenbauer Polynomials

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$$\frac{1}{|\hat{z}-\vec{r}|^{d-2}} = \sum_s r^s\,G_s^{(d)}[\cos\theta]\;;$$

$$G_0^{(d)}(x)=1$$

$$G_1^{(d)}(x)=x$$

$$G_2^{(d)}(x)=dx^2-1$$

$$\vdots$$

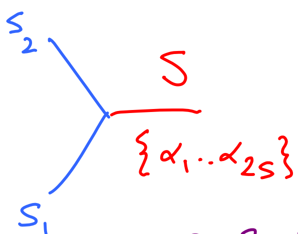
# Life is easy in (3+1) dim!

* Massless little group $\lambda_\alpha \to t\lambda_\alpha$ $\tilde{\lambda}_{\dot\alpha} \to t^{-1}\tilde{\lambda}_{\dot\alpha}$
* Massive little group just SU(2) - easy

$$\epsilon_{\alpha\dot\alpha} \xrightarrow{\left(\frac{p}{m}\right)_\alpha{}^{\dot\beta}} \epsilon_{\alpha\beta}$$

$\{\alpha_1 \ldots \alpha_{2s}\}$ symm. SU(2)

* Naturally chiral, simple action of parity




$\langle 12\rangle[12] = m^2$

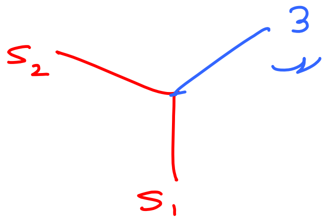
$$g\,\lambda_1^{s+s_2-s_1}\,\lambda_2^{s+s_1-s_2}\,\langle 12\rangle^{-s-s_1-s_2}$$


trivial!

$$\sum_{N_{ij}} \langle 13\rangle^{N_{13}}\langle 14\rangle^{N_{14}}\langle 23\rangle^{N_{23}}\langle 24\rangle^{N_{24}}\,\frac{N_1!\,N_2!\,N_3!\,N_4!}{N_{13}!\,N_{14}!\,N_{23}!\,N_{24}!}$$

$\{N_1 = s + s_2 - s_1 \text{ etc.}\}$

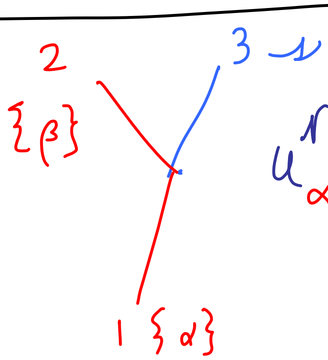
# 2 massive + 1 massless




$$\lambda_\alpha \equiv u_\alpha \,, \quad (p_1 \tilde{\lambda})_\alpha = -(p_2 \tilde{\lambda})_\alpha \equiv v_\alpha$$

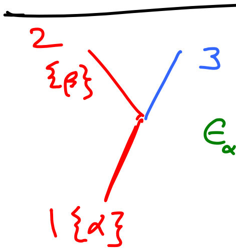
$$\langle u v \rangle = m_1^2 - m_2^2$$

$m_1 \neq m_2$ $\quad u, v$ a basis



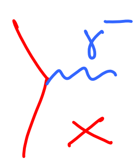

$$u_\alpha^{N_1} \, u_\beta^{N_2} \, v_\alpha^{M_1} \, v_\beta^{M_2}$$

$m_1 = m_2$; $\quad v_\alpha = x\, u_\alpha$

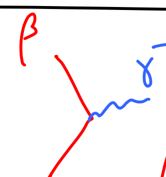



$$\epsilon_{\alpha_1\beta_1} \cdots \epsilon_{\alpha_k\beta_k} \, u^{2(s_1+s_2-k)}_{\alpha\ldots\beta\ldots} \; x^{2[h+k-s_1-s_2]}$$

ex: min coupling EM



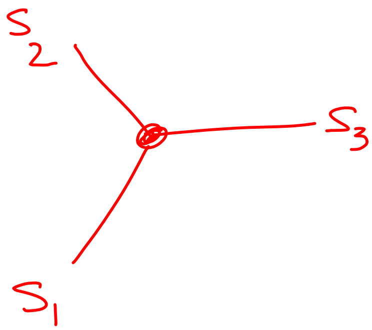

$$x \qquad \propto$$




$$\left( x\, \epsilon_{\alpha\beta} + \frac{\lambda_\alpha \lambda_\beta}{m} \right)$$

charge $\quad g = 2$

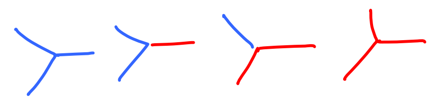
# 3 Massive

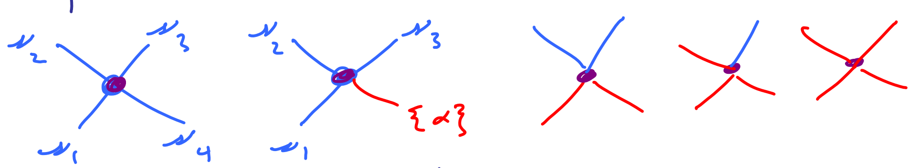



$$\epsilon_{\alpha\beta} \,, \quad (p_i p_j + p_j p_i)_{\alpha\beta}$$

for one of $(ij) = (12), (23), (31)$ provides basis

# Challenge is Concrete

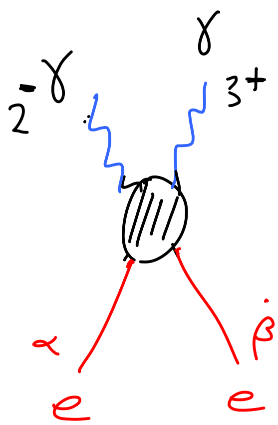
* Given spectrum $\{m_i, \nu_i\}$, couplings 

find $\Big\{$  $\Big\}$

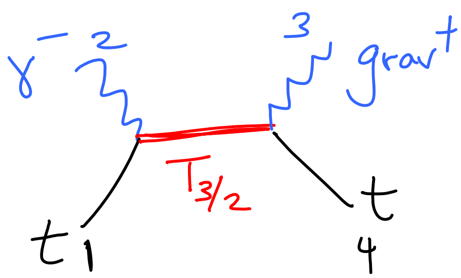


which factorize correctly

# Easy Examples




$$\frac{\lambda_{2\alpha}\,\tilde{\lambda}_{3\dot{\beta}}\,\langle 2|p_4-p_1|3]}{(t-m_e^2)(u-m_e^2)}$$




$$\frac{1}{s-m_T^2}\times\left\{\begin{array}{l} g g'\,\lambda_{2\alpha}\lambda_{3\beta}\langle 2|p_4|3]^3 \\ +\, g g''\,\lambda_{2\alpha}\,p_4|3]_\beta\langle 23\rangle\langle 2|p_4|3]^2 \end{array}\right.$$

$$+\quad 1\leftrightarrow 4 \quad (+\,\text{contact})$$

# Magic of String Positivity

(for massless scattering)

# + Trial Deformations

* Veneziano $\quad A(s,t) = \dfrac{\Gamma(-s-1)\,\Gamma(-t-1)}{\Gamma(-s-t-2)}$

$$\mathrm{Res}[t] \sim \underset{s=-1}{1}\,,\quad \underset{s=0}{(t+2)}\,,\quad \underset{s=1}{(t+2)(t+3)}\,,\ \ldots\,,\ \underset{s=N}{(t+2)\cdots(t+N+2)}$$

$$\underset{s=1}{\mathrm{Res}} = (t+2)(t+3) = \frac{25}{4}\left(\cos^2\theta - \frac{1}{25}\right) = \underbrace{\frac{25}{4}\left(\cos^2\theta - \frac{1}{d}\right)}_{\text{spin 2, OK}} + \underbrace{\frac{25}{4}\left(\frac{1}{d} - \frac{1}{25}\right)}_{\text{spin 0, dangerous!}}$$

OK for $d \lesssim 25$!

* Open superstring $\quad A[1^-2^-3^+4^+] = \langle 12\rangle^2 [34]^2 \dfrac{\Gamma(-s)\,\Gamma(-t)}{\Gamma(1-s-t)}$

$$\mathrm{Res}[t] \sim \underset{s=1}{1}\,,\quad \underset{s=2}{(1+t)}\,,\quad \underset{s=3}{(1+t)(2+t)}\,,\ \ldots\,,\ \underset{s=N}{(1+t)\cdots(N-1+t)}$$

$$\underset{s=3}{\mathrm{Res}} = (t+1)(t+2) = \frac{9}{4}\left(\cos^2\theta - \frac{1}{9}\right) = \frac{9}{4}\left(\cos^2\theta - \frac{1}{d}\right) + \frac{9}{4}\left(\frac{1}{d} - \frac{1}{9}\right)$$

OK for $d \lesssim 9$!

# Positivity at all Levels Miraculous!

$$\left(\cos\theta - \frac{(N-2)}{N}\right)\left(\cos\theta - \frac{(N-4)}{N}\right)\cdots\left(\cos\theta + \frac{(N-2)}{N}\right)$$

$$\underset{\text{analysis}}{\overset{d=2}{=}} \sum_k c_k \cos k\theta \quad \text{with } c_k > 0 \,!$$

* Amazingly simple statement — only known proof is string theory. (NO GHOST THM)

* Hard because $c_k$ become exp small, + adding factor $(\cos\theta - 1)$ makes it false!

# Useful Positive Function Facts

* $\mathcal{F}(x \to 1) > 0$ [ "Positive total X section]
* If $\mathcal{F}$, $G$ positive, $\mathcal{F} + G$, $\mathcal{F} \cdot G$ positive
* If $\mathcal{F}$ is positive in $d$ dimensions, also for $d' < d$.
* $\mathcal{F}(tx) - \mathcal{F}(x)$ is positive for any $t > 1$
* $\mathcal{F}(x)$ has no real roots for $x > 1$

# Open Superstring Positivity is Primary

* Bosonic string residue is $t > 1$ rescaling of open superstring residue

* $\mathrm{Res}^{\mathrm{Grav}}_{N}(\cos\theta) = \left[\mathrm{Res}^{\mathrm{open}}_{N}(\cos\theta)\right]^2$, so, positive

[curiously, "$d_{crit}$" $= 23$]

{ Correction term in het. string not positive, but $\mathrm{Res}^{\mathrm{Grav}}$ "more positive" }

# Trial Deformations

* Trivial : $\sum'$ string amps with different string scales

* "Coon" deformation of Veneziano, $\left[ m_n^2 = \frac{\sigma^n - 1}{\sigma - 1} \right.$ so accumulation point of poles, essential singularity @ finite $s,t$ ]

* Any non-trivial deformations with no essential sing?

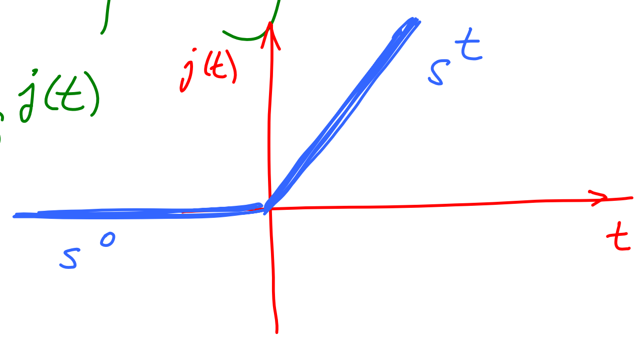
# Toy "Large N QCD" Amps

$$A = (s+t)\,\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1-s-t)} - \left(\frac{1}{s}+\frac{1}{t}\right) = -\zeta(2)(s+t) + \cdots$$

$(s+t) \to \frac{s}{2}(x+1)$ ← positive

pions @ low-E !

* UV high-E fixed angle $\sim \frac{1}{s}$ ; soft, but power-law!
* Infinite tower of string states + Positive
* Regge limit: $s^{j(t)}$




c.f. Komargodski's talk

# More Interesting Examples

$$* \left[\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1-s-t)} - \frac{1}{st}\right] \xrightarrow[t\to t+1]{s\to s+1} \frac{1}{s} + \frac{1}{t} + \cdots \quad \mathrm{Res}^{N}[x] = \underbrace{\mathrm{Res}^{N-4}_{\mathrm{Bos}}[x]}_{\text{Positive}} \underbrace{(1+x)(N-2+Nx)(N+2+Nx)}_{\text{Positive}}$$

UV improve $\varphi^3 \sim \frac{1}{s} \longrightarrow \frac{1}{s^2}$ @ HE

$$* \ \langle 12\rangle^2 [34]^2 \left[\left\{\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1-s-t)} - \frac{1}{st}\right\} \xrightarrow[t\to t+\varepsilon]{s\to s+\varepsilon} + \frac{1}{st}\right]$$

UV improve YM $\sim s^0 \to \frac{1}{s}$ @ HE

# Exponentially Soft YM Deformations?

$* \quad A^{YM} \times (1 + c\,st)$ ← Only polynom. compatible w/ positivity + massless res

$c N^2 \, \mathrm{Res}(x) \left[ x - \left(1 - \frac{1}{cN^2}\right)\right]$ ← last root within $1/N^2$ of 1

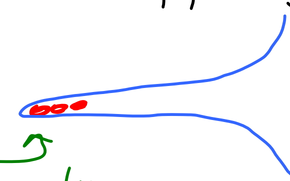
Empirically: KILLS POSITIVITY

$$* \quad \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1-s-t)} = \frac{1}{st}\int_0^1 dx\, x^{-s}\partial_x (1-x)^{-t} \longrightarrow \frac{1}{st}\int_0^1 dx\, f(x)^{-s}\partial_x f(1-x)^{-t}$$

$$f(x) = x + \varepsilon x^k (1-x)^m \Longrightarrow \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1-s-t)} + \varepsilon \frac{\Gamma(1-s)\Gamma(1-t)}{\Gamma(2-s-t)} + \cdots$$

{But truncation easily excluded — at level $\sim 1/\epsilon$ root $> 1$}

* There should exist YM amps that are exp. soft in UV + differ from pert strings e.g.




* But not continuous deform. of pert string amp.

Is $\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1-s-t)}$ locally unique exp soft YM amp?

* Note: no known warped compactifications with param. weak coupling — gravity should be unique!

# Exponentially Soft GR Deformation

$$\frac{\Gamma(-s)\Gamma(-t)\Gamma(-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} + \varepsilon\frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(2+s)\Gamma(2+t)\Gamma(2+u)} + \cdots$$

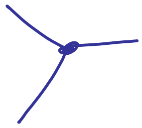
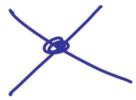
$$\mathrm{Res}_N^{\varepsilon}[x] = \left[1-\varepsilon+\frac{1}{N}\right]\left\{\mathrm{Res}_N^{\varepsilon=0}[x] + \frac{4\varepsilon(N-1)}{N(1+(1-\varepsilon)N)}\underbrace{\mathrm{Res}_N^{\text{open}}(x)}_{\text{pos}}\,\underbrace{\mathrm{Res}_{N-4}^{\text{bos}}(x)}_{\text{pos}}\right\}$$

ok for $0 \leq \epsilon \leq 1$

# Lessons

* Positivity of string amps is pure magic!
* But, with just massless scattering, some deformations inheriting this magic exist — very unlikely to be consistent theories
* Indication we must look at consistency of amps involving massive higher spin particles

# Next Steps + Outlook

* Extend   rules to general D [+SUSY]

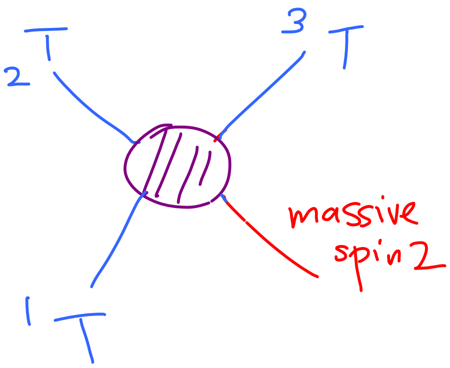
* Prove $\left(\cos\theta - \frac{(N-2)}{N}\right) \cdots \left(\cos\theta + \frac{(N-2)}{N}\right) = \sum_k c_k \cos k\theta$, $c_k \geq 0$ in a new way! that explains....

* Why it breaks if we $\times \left(\cos\theta - 1 + \frac{c}{N^2}\right)$

* Can we find/rule out exp soft YM def?

* What higher-spin massive amps exclude [?] our toy "large-N pion" amps, deformed gravity amps?

# A Peek @ Massive Amps




$$\frac{\Gamma(-s-1)\Gamma(-t-1)}{\Gamma(u)}\left[(\epsilon p_1 p_1) - 2(\epsilon p_1 p_2)\frac{t+1}{u} + \epsilon(p_2 p_2)\frac{t(t+1)}{u(u+1)}\right]$$

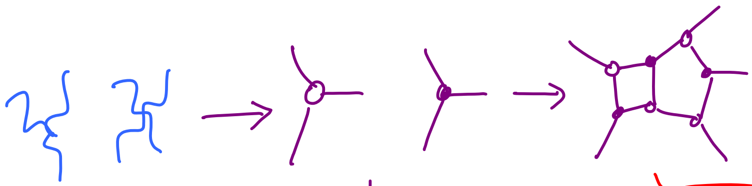
Decompose into pol. structures $\rightsquigarrow$

$$\begin{aligned}
a_1 &: \prod_{i=1}^{N-1}\left(x - \frac{N-2i}{\sqrt{(4+N)(4+N^2)/N}}\right)\\
a_2 &: x\prod_{i=1}^{N-1}\left(x - \frac{N-2i}{\sqrt{(4+N)(4+N^2)/N}}\right)\\
a_3 &: x\prod_{i=1}^{N-1}\left(x - \frac{N+2-2i}{\sqrt{(4+N)(4+N^2)/N}}\right),
\end{aligned}$$

Closer to going negative (but still pos!) than tachyon amps

Obviously the real goal isn't the academic exercise of "deriving pert. strings". Really we want to understand pert. strings in a new way, where the worldsheet picture isn't primary

# (S)YM



Gauge-redundant $\mathcal{L}$ → + 4pt check → Positive Grassmannian ← Amplituhedron

Yangian manifest {Fermionic T-duality}

Amplituhedron ↔ {Dual Amplituhedron}

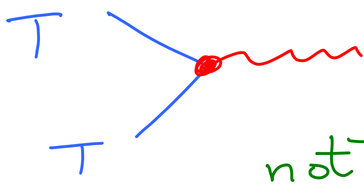
Might there be a similar story for pert. string theory?

(At any rate — Obvious next step in Amplitudes program)

# We need more Data!

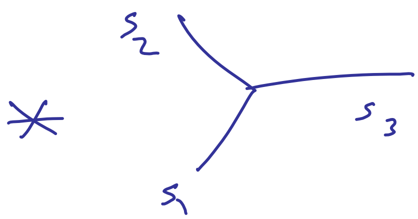
For greatest S-matrix theory of all time, our understanding of string amps is very primitive...

e.g. Covariant bosonic string spectrum not explicitly known!




$g \in (p_1-p_2)\cdots(p_1-p_2)$ {never mind all the rest!}

not known!




fixed by Poincaré; what principle fixes the string 3pt amps?