Bootstrapping the 3D Ising Model

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The Conformal Bootstrap

Polyakov '70: classify/solve CFTs using:

- conformal symmetry
- unitarity
- associativity of the OPE

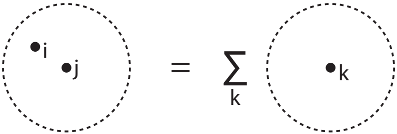
Progress in $d = 2$ throughout 80's and 90's.

Huge revival for $d > 2$ a few years ago...

CFT Review

- Local operators $\mathcal{O}_1(x)$, $\mathcal{O}_2(x)$, ...
- Scaling dimensions $\langle \mathcal{O}_i(x)\mathcal{O}_i(y)\rangle = |x-y|^{-2\Delta_i}$
- Operator Product Expansion (OPE)

$$\mathcal{O}_i(x)\mathcal{O}_j(0) \;=\; \sum_k f_{ijk}\, x^{\Delta_k-\Delta_i-\Delta_j}\,(\mathcal{O}_k(0)+\dots)$$



- Unitarity: Δ_i bounded from below, f_{ijk} are real

Bootstrap Revival

- $\phi(x)$: a real scalar primary operator.
- It has the OPE

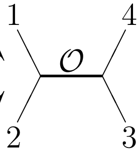
$$\phi(x)\phi(0) \;=\; \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}}\, x^{\Delta_{\mathcal{O}}-2\Delta_\phi}\,(\mathcal{O}(0)+\ldots)$$

Rattazzi, Rychkov, Tonni, Vichi '08: Bootstrap constraints on $\langle\phi\phi\phi\phi\rangle$ imply universal bounds on

- OPE coefficients $f_{\phi\phi\mathcal{O}}$
- Dimensions, spins $\Delta_{\mathcal{O}}, \ell_{\mathcal{O}}$

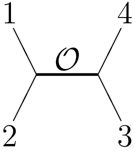
Conformal Blocks & Crossing Symmetry

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{\mathcal{O}}$$



Crossing Symmetry

$$\sum_{\mathcal{O}} \Bigg($$



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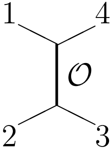


$$\Bigg) = 0$$

$$\sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} \underbrace{\left(v^{\Delta_\phi} g_{\Delta,\ell}(u,v) - u^{\Delta_\phi} g_{\Delta,\ell}(v,u)\right)}_{F_{\Delta,\ell}(u,v)} = 0$$

Bounds from Crossing Symmetry

$$0 \;=\; F_{0,0}(u,v) + \sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} F_{\Delta,\ell}(u,v)$$

- Make an assumption about spectrum of Δ, ℓ's.
- Try to find a linear functional α such that

$$\begin{aligned} \alpha(F_{0,0}) &\;>\; 0 \\ \alpha(F_{\Delta,\ell}) &\;\geq\; 0 \end{aligned}$$

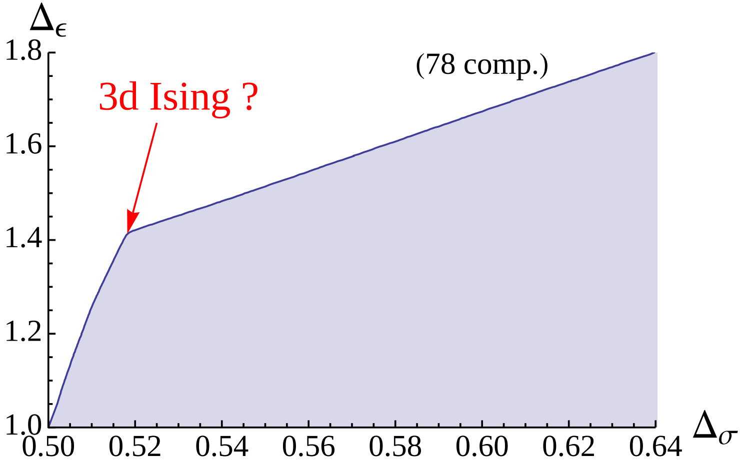
  (convex optimization problem)
- If α exists, assumption is ruled out.

Outline

1. Bounds in 3d CFTs
2. Mixed Correlators
3. Future Directions

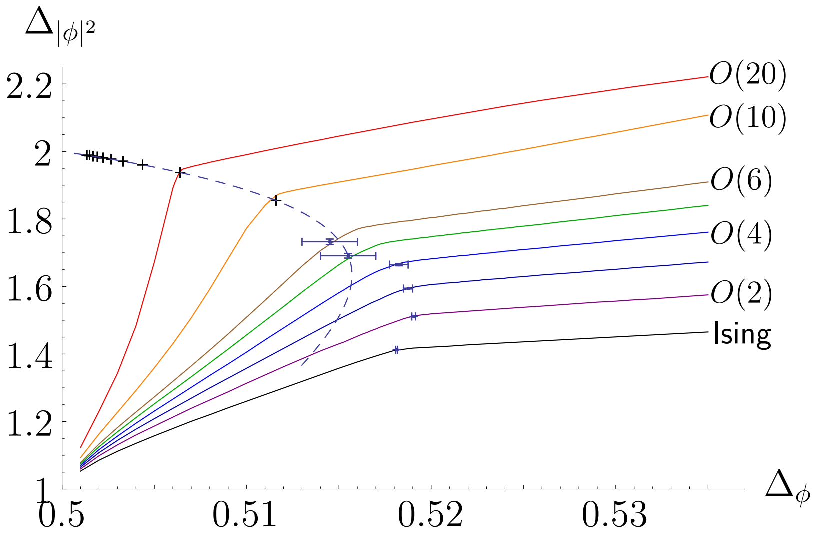
Outline

Universal Bound in 3d CFTs [El-Showk, Paulos, Poland, Rychkov, DSD, Vichi '12]



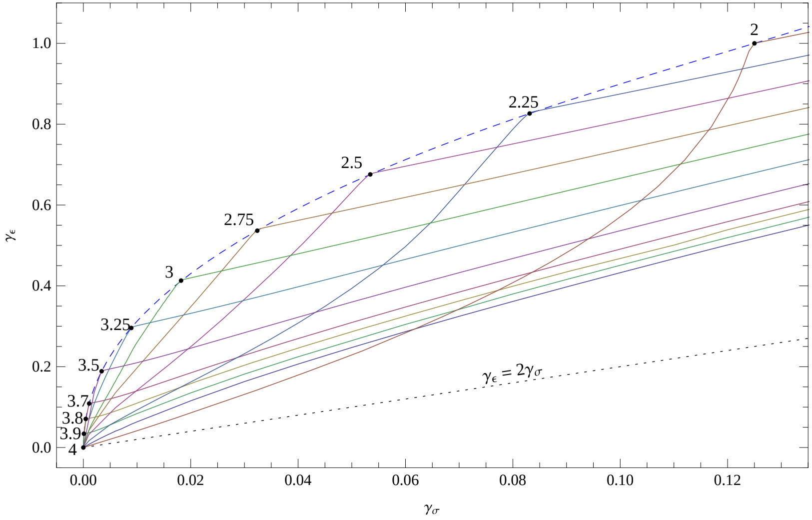
- $\epsilon \equiv$ lowest dimension scalar in $\sigma \times \sigma$
- Assumes only bootstrap constraints for $\langle \sigma\sigma\sigma\sigma \rangle$

3d $O(N)$ Vector Models [Kos, Poland, DSD '13]



Fractional Spacetime Dimensions [El-Showk, Paulos, Poland, Rychkov, DSD, Vichi '13]

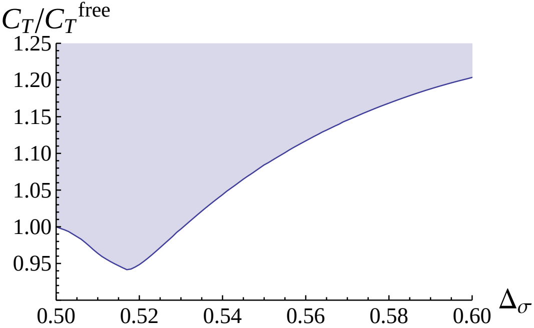
$$\gamma_\epsilon \equiv \Delta_\epsilon - (d-2) \quad \text{vs.} \quad \gamma_\sigma \equiv \Delta_\sigma - \frac{d-2}{2}$$



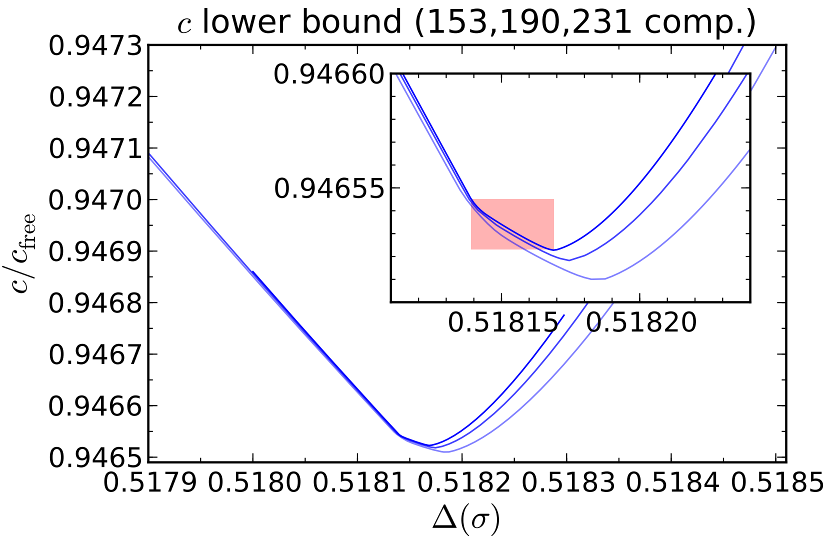
c-Minimization

- Perhaps $\langle \sigma\sigma\sigma\sigma \rangle$ in 3d Ising lies on the boundary of the space of unitary, crossing-symmetric 4-pt functions.

Natural conjecture: Ising minimizes $c \propto \langle T_{\mu\nu} T_{\rho\sigma} \rangle$
[El-Showk, Paulos, Poland, Rychkov, DSD, Vichi '14]



c at High Precision



Spectrum from c-Minimization [El-Showk, Paulos, Poland, Rychkov, DSD, Vichi '14]

year	Method	ν	η	ω
1998	ϵ-exp	0.63050(250)	0.03650(500)	0.814(18)
1998	3D exp	0.63040(130)	0.03350(250)	0.799(11)
2002	HT	0.63012(16)	0.03639(15)	0.825(50)
2003	MC	0.63020(12)	0.03680(20)	0.821(5)
2010	MC	0.63002(10)	0.03627(10)	0.832(6)
	c-min	0.62999(5)	0.03631(3)	0.8303(18)

Critical exponents:

$$\Delta_\sigma = 1/2 + \eta/2, \quad \Delta_\epsilon = 3 - 1/\nu, \quad \Delta_{\epsilon'} = 3 + \omega\,.$$

Outline

Mixed Correlators [Kos, Poland, DSD '14]

- So far, bootstrap studies have focused on 4-pt function of identical operators $\langle\phi\phi\phi\phi\rangle$.
- Full bootstrap requires crossing-symmetry & unitarity for all 4-pt functions.
- Mixed correlator: $\langle\sigma\sigma\epsilon\epsilon\rangle$ in 3d Ising.
- Consequences of unitarity are trickier:

$$\langle\sigma\sigma\epsilon\epsilon\rangle \;=\; \sum_{\mathcal{O}} f_{\sigma\sigma\mathcal{O}} f_{\epsilon\epsilon\mathcal{O}} g_{\Delta,\ell}(u,v)$$

$f_{\sigma\sigma\mathcal{O}} f_{\epsilon\epsilon\mathcal{O}}$ not necessarily positive.

Positivity for Mixed Correlators

- Consider $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\sigma\sigma\epsilon\epsilon\rangle$, $\langle\epsilon\epsilon\epsilon\epsilon\rangle$ together. Crossing symmetry says:

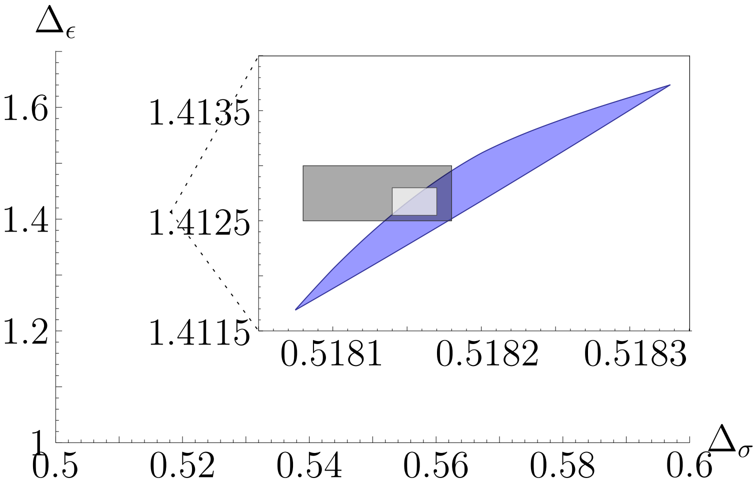
$$\sum_{\mathcal{O}} \begin{pmatrix} f_{\sigma\sigma\mathcal{O}} & f_{\epsilon\epsilon\mathcal{O}} \end{pmatrix} \begin{pmatrix} F^{(1,1)}_{\Delta,\ell}(u,v) & F^{(1,2)}_{\Delta,\ell}(u,v) \\ F^{(2,1)}_{\Delta,\ell}(u,v) & F^{(2,2)}_{\Delta,\ell}(u,v) \end{pmatrix} \begin{pmatrix} f_{\sigma\sigma\mathcal{O}} \\ f_{\epsilon\epsilon\mathcal{O}} \end{pmatrix} + \cdots = 0$$

- Look for functionals $\alpha : F(u,v) \to \mathbb{R}$ such that

$$\begin{pmatrix} \alpha(F^{(1,1)}_{\Delta,\ell}) & \alpha(F^{(1,2)}_{\Delta,\ell}) \\ \alpha(F^{(2,1)}_{\Delta,\ell}) & \alpha(F^{(2,2)}_{\Delta,\ell}) \end{pmatrix} \succeq 0$$

is positive semidefinite. Analog of $\alpha(F_{\Delta,\ell}) \geq 0$.

Mixed Correlator Bound for CFT_3 w/ $\mathbb{Z}_2$



- Monte-Carlo, c-min conjecture, rigorous bound
- Assuming σ, ϵ are only relevant scalars.

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Future Directions

- Improve optimization algorithms/precision
- Find more boundary-dwelling CFTs ([3d, 5d: Nakayama, Ohtsuki] [4d $\mathcal{N}=2,4$, 6d $\mathcal{N}=(2,0)$: Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees] [4d $\mathcal{N}=4$ Alday, Bissi] [3d $\mathcal{N}=8$: Chester, Lee, Pufu, Yacoby])
- Mixed correlators in other theories
- Four-point functions of operators with spin (stress tensor, symmetry currents)
- Nonlocal operators [Liendo, Rastelli, van Rees '12] [Gaiotto, Mazac, Paulos '13]
- Analytic results, new consistency conditions