Exact results and modular invariance of integrated correlators in $\mathcal{N} = 4$ SYM

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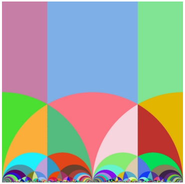
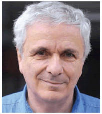
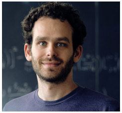
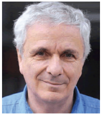


Introduction

- Based on arXiv: 2102.09537, arXiv: 2102.08305,
  with Daniele Dorigoni, Michael Green

- Early work, arXiv: 1912.13365, arXiv: 2008.02713,
  with Shai Chester, Michael Green, Silviu Pufu, Yifan Wang

A four-point correlator in $SU(N)$ $\mathcal{N}=4$ SYM

- We will study four-point correlator of Chiral Primary Operators,

$$\mathcal{O}_2(x,Y) = \mathrm{tr}(\phi_{I_1}(x)\phi_{I_2}(x))Y^{I_1}Y^{I_2}\,,$$

where $I_p = 1, 2, \cdots, 6$ and $Y \cdot Y = 0$.

- Two- and three-point correlators are protected.
- Supersymmetry and superconformal symmetries imply [Eden, Petkou, Schubert, Sokatchev][Nirschl, Osborn]

$$\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle = \langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle_{\rm free} + \mathcal{I}_4(x_i, Y_i)\,\mathcal{T}_N(U,V;\tau,\bar{\tau})\,,$$

where $\mathcal{I}_4$ is fixed by the symmetries and we focus on $\mathcal{T}_N$.
U, V are cross ratios & $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2_{\rm YM}} = \tau_1 + i\tau_2$.

A four-point correlator in $SU(N)$ $\mathcal{N} = 4$ SYM

What is known about the correlator?

- Weak coupling expansion:
  - Known up to 3 loops [Drummond, Duhr, Eden, Heslop, Pennington, Smirnov].
  - In planar limit, the integrand was constructed up to 10 loops [Bourjaily, Heslop, Tran].
  - The first non-planar contribution enters at 4 loops [Fleury, Pereira].
- Strong coupling expansion can be computed using Witten diagrams [D'Hoker, Freedman, Mathur, Matusis, Rastelli]...; more recently: KK modes, loop corrections, string corrections ... [Rastelli, Zhou][Alday, Bissi, + Perlmutter][Aprile, Drummond, Heslop, Paul][Alday, Zhou][Bissi, Fardelli, Georgoudis][Drummond, Paul][Alday, Caron-Huot][Caron-Huot, Trinh][Aprile, Vieira][Abl, Heslop, Lipstein]....
- Instanton effects were studied in the semi-classical limit. [Bianchi, Green, Kovacs, Rossi][Dorey, Hollowood, Khoze, Mattis, Vandoren]...

Integrated correlators in $SU(N)$ $\mathcal{N}=4$ SYM

- We are interested in $SL(2,\mathbb{Z})$ modular properties and the correlator at finite coupling τ.
- This in general is very difficult; we will consider a simpler yet highly non-trivial object: integrated correlators,

$$\mathcal{G}_N(\tau,\bar{\tau}) = \int dUdV\, M(U,V)\, \mathcal{T}_N(U,V;\tau,\bar{\tau})\,.$$

  With suitable choices of the measure to preserve supersymmetry, $\mathcal{G}_N(\tau,\bar{\tau})$ can be computed exactly.
- One may reconstruct the un-integrated correlator at finite coupling, at least for first few orders in large-N expansion.

Integrated correlators in $SU(N)$ $\mathcal{N}=4$ SYM

Two integrated correlators have been studied.

- Integrated correlator no. 1: [Binder, Chester, Pufu, Wang] [Chester, Pufu]

$$\mathcal{G}_{1N}(\tau,\bar{\tau}) = -\frac{8}{\pi}\int_0^\infty dr \int_0^\pi d\theta \, \frac{r\sin^2(\theta)}{U} \mathcal{T}_N(U,V;\tau,\bar{\tau})\,,$$

with $U = 1 + r^2 - 2r\cos(\theta)$, $V = r^2$.

- Integrated correlator no. 2: [Chester, Pufu]

$$\mathcal{G}_{2N}(\tau,\bar{\tau}) = -\frac{96}{\pi}\int_0^\infty dr \int_0^\pi d\theta \, \frac{r\sin^2(\theta)}{U} \bar{D}_{1111}(U,V)\mathcal{T}_N(U,V;\tau,\bar{\tau})\,,$$

where $\bar{D}_{1111}(U,V)$ is the 1-loop box.

Integrated correlators in $SU(N)$ $\mathcal{N}=4$ SYM

Simplicity of the integrated correlators. E.g., for $\mathcal{G}_{1N}(\tau,\bar{\tau})$.

- 1 and 2 loops: the correlator is given by ladder diagrams [Usyukina, Davydychev]

$$f^{(L)}(z,\bar{z}) = \sum_{r=0}^{L} \frac{(-1)^r(2L-r)!}{r!(L-r)!L!} \log^r(z\,\bar{z})\,(\mathrm{Li}_{2L-r}(z) - \mathrm{Li}_{2L-r}(\bar{z}))\ ,$$

  whereas the integrated L-loop ladder diagram is simply [Usyukina]

$$-2\binom{2L+2}{L+1}\zeta(2L+1)\,.$$

- 3 loops: given by a pages-long expression involving multiple polylogarithms, however the integrated result is simply $[g_{\mathrm{YM}}^2 N/(4\pi^2)]^3\, 735/16\,\zeta(7)$.

Integrated correlators from localization

The integrated correlators can be computed exactly.

- They are determined by four derivatives of the partition function of $\mathcal{N}=2^*$ SYM on S^4 [Binder, Chester, Pufu, Wang] [Chester, Pufu]

$$\mathcal{G}_{1N}(\tau,\bar\tau) = \tau_2^2 \partial_\tau \partial_{\bar\tau} \partial_m^2 \log Z_N(m,\tau,\bar\tau)\big|_{m=0}\,,$$
$$\mathcal{G}_{2N}(\tau,\bar\tau) = \partial_m^4 \log Z_N(m,\tau,\bar\tau)\big|_{m=0}\,,$$

where $Z_N(m,\tau,\bar\tau)$ is computed using supersymmetric localization [Nekorasov][Pestun]...

$$Z_N(m,\tau,\bar\tau) = \int d^N a\, \delta(\sum_i a_i) \prod_{i<j} (a_i - a_j)^2\, e^{-\frac{8\pi^2}{g^2_{\rm YM}} \sum_i a_i^2}\, Z_{\rm pert}\, |Z_{\rm inst}|^2\,.$$

- We will mostly focus on $\mathcal{G}_{1N}(\tau,\bar\tau)$ & drop "1", and briefly discuss $\mathcal{G}_{2N}(\tau,\bar\tau)$ at the end.

Exact results of an integrated correlator

By carefully analysing $\mathcal{N} = 2^*$ SYM partition function, we conjectured an exact expression for $\mathcal{G}_{1N}(\tau, \bar{\tau})$ for arbitrary N and τ:

$$\mathcal{G}_N(\tau, \bar{\tau}) = \sum_{(p,q)\in\mathbb{Z}^2} \int_0^\infty \exp\left(- t\pi \frac{|p + q\tau|^2}{\tau_2}\right) B_N(t)\, dt\,,$$

where $B_N(t) = \frac{\mathcal{Q}_N(t)}{(t+1)^{2N+1}}$, & $\mathcal{Q}_N(t)$ is a degree-$(2N-1)$ polynomial:

$$\mathcal{Q}_N(t) = -\frac{1}{4} N(N-1)(1-t)^{N-1}(1+t)^{N+1}$$
$$\left\{(3 + (8N + 3t - 6)\, t)\, P_N^{(1,-2)}(z) + \frac{3t^2 - 8Nt - 3}{t+1} P_N^{(1,-1)}(z)\right\}\,,$$

with $z = \frac{1+t^2}{1-t^2}$, $P_N^{(\alpha,\beta)}$ is the Jacobi polynomial. E.g.

$$\mathcal{Q}_2(t) = 9t^3 - 30t^2 + 9t\,,$$
$$\mathcal{Q}_3(t) = 18t^5 - 99t^4 + 126t^3 - 99t^2 + 18t\,.$$

Exact results of an integrated correlator

Some remarks:

- $B_N(t) = 1/t B_N(1/t)$, $\int_0^\infty B_N(t) dt/\sqrt{t} = 0$, ...
- k-instanton term $e^{2\pi i k \tau_1}$ has $k = \hat{p}q$, where $\hat{p}$ replaces p via the Poisson sum.
- $\mathcal{G}_N(\tau, \bar{\tau})$ is manifestly $SL(2,\mathbb{Z})$ invariant.
- Formally $\mathcal{G}_N(\tau, \bar{\tau})$ can be re-expressed as an infinite sum:

$$\mathcal{G}_N(\tau, \bar{\tau}) = \sum_{s=2}^{\infty} c_N(s) E(s; \tau, \bar{\tau}) \,.$$

The non-holomorphic Eisenstein series

$$E(s; \tau, \bar{\tau}) = \sum_{(p,q)\neq(0,0)} \frac{\tau_2^s}{\pi^s |p + q\tau|^{2s}}$$

$$= \frac{2\zeta(2s)}{\pi^s} \tau_2^s + \frac{2\zeta(2s-1)\Gamma(s-\frac{1}{2})}{\pi^{s-\frac{1}{2}}\Gamma(s)} \tau_2^{1-s} + \text{instantons} \,.$$

Exact results of an integrated correlator

- $B_N(t)$ obeys a differential equation, that leads to a $SL(2,\mathbb{Z})$ invariant Laplace-difference equation for $\mathcal{G}_N(\tau,\bar\tau)$,

$$\left(4\tau_2^2\partial_\tau\partial_{\bar\tau}-2\right)\mathcal{G}_N(\tau,\bar\tau)=N^2\left[\mathcal{G}_{N+1}(\tau,\bar\tau)-2\mathcal{G}_N(\tau,\bar\tau)+\mathcal{G}_{N-1}(\tau,\bar\tau)\right]$$
$$-N\left[\mathcal{G}_{N+1}(\tau,\bar\tau)-\mathcal{G}_{N-1}(\tau,\bar\tau)\right].$$

  - As a comparison: the non-holomorphic Eisenstein series obeys a homogeneous Laplace equation

$$\left[4\tau_2^2\partial_\tau\partial_{\bar\tau}-s(s-1)\right]E(s;\tau,\bar\tau)=0\,.$$

  - $\mathcal{G}_1(\tau,\bar\tau)=0$. Once $\mathcal{G}_2(\tau,\bar\tau)$ is given, the Laplace-difference equation determines $\mathcal{G}_N(\tau,\bar\tau)$ for all N.

- We will now study $\mathcal{G}_N(\tau,\bar\tau)$ in various limits.

Weak-coupling perturbative expansion

Weak-coupling perturbative expansion (loops)

$$\mathcal{G}_{N,0}(\tau_2) = 4c\left[\frac{3\,\zeta(3)a}{2} - \frac{75\,\zeta(5)a^2}{8} + \frac{735\,\zeta(7)a^3}{16} - \frac{6615\,\zeta(9)\left(1+\frac{2}{7}N^{-2}\right)a^4}{32} + \frac{114345\,\zeta(11)\left(1+N^{-2}\right)a^5}{128} - \frac{3864861\,\zeta(13)\left(1+\frac{25}{11}N^{-2}+\frac{4}{11}N^{-4}\right)a^6}{1024} + \cdots\right],$$

with $a = \lambda/(4\pi^2)$ and $4c = N^2 - 1$.

- It gives an all-loop prediction for any N.
- 1-, 2- and 3-loop terms were proved to agree with known results.
- Non-planar contributions start to enter at 4 loops, in agreement with known results.

Large N: small-λ expansion

Large-N expansion: $\mathcal{G}_N(\tau,\bar\tau) \sim \sum_{g=0}^{\infty} N^{2-2g}\,\mathcal{G}^{(g)}(\lambda)$.

- Small-λ expansion

$$\mathcal{G}^{(0)}(\lambda) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}\zeta(2n+1)\Gamma\left(n+\frac{3}{2}\right)^2}{\pi^{2n+1}\Gamma(n)\Gamma(n+3)}\lambda^n\,,$$

$$\mathcal{G}^{(1)}(\lambda) = \sum_{n=1}^{\infty} \frac{(-1)^n(n-5)(2n+1)\zeta(2n+1)\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{24\,\pi^{2n+1}\Gamma(n)^2}\lambda^n\,,$$

$$\vdots$$

- They are all convergent with a finite radius $|\lambda| < \pi^2$, which has been seen in $\mathcal{N}=4$ SYM, such as cusp anomalous dimension [Basso, Korchemsky, Kotanski], amplitudes [Basso, Dixon, Papathanasiou].

Large N: large-λ expansion

- Large-λ expansion:

$$\mathcal{G}^{(0)}(\lambda) \sim \frac{1}{4} + \sum_{n=1}^{\infty} \frac{\Gamma\left(n-\frac{3}{2}\right)\Gamma\left(n+\frac{3}{2}\right)\Gamma(2n+1)\zeta(2n+1)}{2^{2n-2}\pi\,\Gamma(n)^2\,\lambda^{n+1/2}},$$

$$\mathcal{G}^{(1)}(\lambda) \sim -\frac{\sqrt{\lambda}}{16} - \sum_{n=1}^{\infty} \frac{n^2(2n+11)\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)^2\zeta(2n+1)}{24\,\pi^{\frac{3}{2}}\Gamma(n+2)\,\lambda^{n+1/2}},$$

$$\vdots$$

- They are all asymptotic & not Borel summable, require non-perturbative completions

$$\Delta\mathcal{G}^{(0)}(\lambda) = i\left[8\mathrm{Li}_0(e^{-2\sqrt{\lambda}}) + \frac{18\mathrm{Li}_1(e^{-2\sqrt{\lambda}})}{\lambda^{1/2}} + \frac{117\mathrm{Li}_2(e^{-2\sqrt{\lambda}})}{4\lambda} + \cdots\right],$$

$$\Delta\mathcal{G}^{(1)}(\lambda) = i\left[-\frac{127\mathrm{Li}_0(e^{-2\sqrt{\lambda}})}{2^8} + \frac{927\mathrm{Li}_1(e^{-2\sqrt{\lambda}})}{2^{12}\lambda^{1/2}} - \frac{3897\mathrm{Li}_2(e^{-2\sqrt{\lambda}})}{2^{14}\lambda} + \cdots\right],$$

$$\vdots$$

Large N: finite YM coupling τ

Large-N expansion with finite Yang-Mills coupling τ ("very strong coupling limit"):

$$\begin{aligned}
\mathcal{G}_N(\tau,\bar\tau) \sim{}& \frac{N^2}{4} - \frac{3N^{\frac{1}{2}}}{2^4} E(\tfrac{3}{2};\tau,\bar\tau) + \frac{45}{2^8 N^{\frac{1}{2}}} E(\tfrac{5}{2};\tau,\bar\tau) \\
&+ \frac{3}{N^{\frac{3}{2}}}\Big[\frac{1575}{2^{15}} E(\tfrac{7}{2};\tau,\bar\tau) - \frac{13}{2^{13}} E(\tfrac{3}{2};\tau,\bar\tau)\Big] + \frac{225}{N^{\frac{5}{2}}}\Big[\frac{441}{2^{18}} E(\tfrac{9}{2};\tau,\bar\tau) - \frac{5}{2^{16}} E(\tfrac{5}{2};\tau,\bar\tau)\Big] \\
&+ \frac{63}{N^{\frac{7}{2}}}\Big[\frac{3898125}{2^{27}} E(\tfrac{11}{2};\tau,\bar\tau) - \frac{44625}{2^{25}} E(\tfrac{7}{2};\tau,\bar\tau) + \frac{73}{2^{22}} E(\tfrac{3}{2};\tau,\bar\tau)\Big] \\
&+ \frac{945}{N^{\frac{9}{2}}}\Big[\frac{31216185}{2^{31}} E(\tfrac{13}{2};\tau,\bar\tau) - \frac{41895}{2^{26}} E(\tfrac{9}{2};\tau,\bar\tau) + \frac{1639}{2^{27}} E(\tfrac{5}{2};\tau,\bar\tau)\Big] + \cdots .
\end{aligned}$$

Recall $E(s;\tau,\bar\tau)$ is the non-holomorphic Eisenstein series, which is $SL(2,\mathbb{Z})$ invariant.

Integrated correlator no. 2 & new modular invariants

Integrated correlator no. 2 at finite coupling τ, up to $1/N^3$:

$$\partial_m^4 \log Z|_{m=0} = 6N^2 + 6N^{\frac{1}{2}} E(\tfrac{3}{2};\tau,\bar\tau) + C_0 - \frac{9}{2N^{\frac{1}{2}}} E(\tfrac{5}{2};\tau,\bar\tau) - \frac{27}{2^3 N}\mathcal{E}(3,\tfrac{3}{2},\tfrac{3}{2};\tau,\bar\tau)$$

$$-\frac{9}{N^{\frac{3}{2}}}\left[\frac{375}{2^{10}} E(\tfrac{7}{2};\tau,\bar\tau) - \frac{13}{2^8} E(\tfrac{3}{2};\tau,\bar\tau)\right] + \frac{405}{704 N^2}\left[C_1 + 35\mathcal{E}(6,\tfrac{5}{2},\tfrac{3}{2};\tau,\bar\tau)\right.$$

$$\left. - 24\mathcal{E}(4,\tfrac{5}{2},\tfrac{3}{2};\tau,\bar\tau)\right] - \frac{675}{N^{\frac{5}{2}} 2^{10}}\left[\frac{49}{4} E(\tfrac{9}{2};\tau,\bar\tau) - E(\tfrac{5}{2};\tau,\bar\tau)\right]$$

$$+\frac{1}{N^3}\Big[\alpha_3 \mathcal{E}(3,\tfrac{3}{2},\tfrac{3}{2};\tau,\bar\tau) + \sum_{r=5,7,9}\big[\alpha_r \mathcal{E}(r,\tfrac{3}{2},\tfrac{3}{2};\tau,\bar\tau) + \beta_r \mathcal{E}(r,\tfrac{5}{2},\tfrac{5}{2};\tau,\bar\tau) + \gamma_r \mathcal{E}(r,\tfrac{7}{2},\tfrac{3}{2};\tau,\bar\tau)\big]\Big],$$

where $\alpha_i, \beta_i, \gamma_i$ are rational numbers; $\mathcal{E}$ is the generalised non-holomorphic Eisenstein series

$$[4\tau_2 \partial_\tau \partial_{\bar\tau} - r(r+1)]\, \mathcal{E}(r, s_1, s_2;\tau,\bar\tau) = -E(s_1;\tau,\bar\tau) E(s_2;\tau,\bar\tau)\,.$$

Type IIB string amplitudes in $\mathrm{AdS}_5 \times S^5$

- To reconstruct the un-integrated correlator, we write an ansatz for it, that is conveniently done in Mellin space [Mack][Penedones]

$$\mathcal{T}_N(U,V;\tau,\bar\tau) = \int \frac{ds\,dt}{(4\pi i)^2} U^{\frac{s}{2}} V^{\frac{t-4}{2}} \Gamma^2\!\left(\frac{4-s}{2}\right)\Gamma^2\!\left(\frac{4-t}{2}\right)\Gamma^2\!\left(\frac{s+t-4}{2}\right)\mathcal{M}_N(s,t;\tau,\bar\tau)\,.$$

- Large-N ansatz (after removing an overall R^4),

$$\mathcal{M}(s,t;\tau,\bar\tau) = \frac{a\,c}{(s-2)(t-2)(u-2)} + c^{1/4} b + \mathcal{M}^{\mathrm{SUGRA}}_{\text{1-loop}}(s,t) + \frac{c_2(s^2+t^2+u^2)+c_1}{c^{1/4}} + \frac{d_3 stu + d_2(s^2+t^2+u^2)+d_1}{c^{1/2}} + \cdots\,.$$

- Unknown coefficients are fixed by two integrated correlators, and the flat-space limit in the case of d^6R^4.

Type IIB string amplitudes in $\mathrm{AdS}_5 \times S^5$

The exact result of the un-integrated correlator in large-N expansion (after removing an overall R^4),

$$\begin{aligned}\mathcal{M}(s,t;\tau,\bar\tau) &= \frac{8c}{(s-2)(t-2)(u-2)} + \frac{15E(\tfrac{3}{2};\tau,\bar\tau)c^{1/4}}{4\sqrt{2\pi^3}} + \mathcal{M}^{\mathrm{SUGRA}}_{\text{1-loop}}(s,t)\\ &+ \frac{315E(\tfrac{5}{2};\tau,\bar\tau)}{128\sqrt{2\pi^5}c^{1/4}}\left[(s^2+t^2+u^2)-3\right]\\ &+ \frac{945\,\mathcal{E}(3,\tfrac{3}{2},\tfrac{3}{2};\tau,\bar\tau)}{64\pi^3c^{1/2}}\left[stu-\frac{1}{4}(s^2+t^2+u^2)-4\right]+\cdots\end{aligned}$$

In flat-space limit, $\mathcal{M}(s,t;\tau,\bar\tau)$ reproduces known results of superstring amplitudes in flat space. [Green, Gutperle + Vanhove][Green, Sethi] ...

$$\begin{aligned}\mathcal{L}^{\mathrm{IIB}}_{\mathrm{EFT}} &\sim \alpha'^{-4}R + \alpha'^{-1}E(\tfrac{3}{2};\tau,\bar\tau)R^4 + \alpha' E(\tfrac{5}{2};\tau,\bar\tau)d^4R^4\\ &+ \alpha'^2\mathcal{E}(3,\tfrac{3}{2},\tfrac{3}{2};\tau,\bar\tau)d^6R^4+\cdots.\end{aligned}$$

Summary and comments

- The integrated correlators can be computed exactly, and provide tools for studying non-perturbative effects.
- Integrated correlator $\partial_m^4 \log Z$ at finite N & finite τ?
- Higher-point bonus $U(1)_Y$-violating correlators: non-holomorphic modular forms. [Green, C.W.]
- Correlators of higher-weight Chiral Primary Operators: Hidden 10d conformal symmetry.[see the talk by Coronado.]

Thank you!