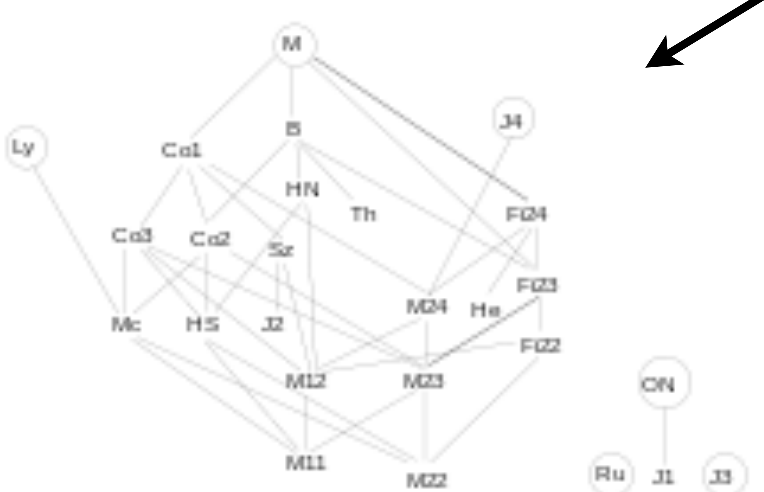
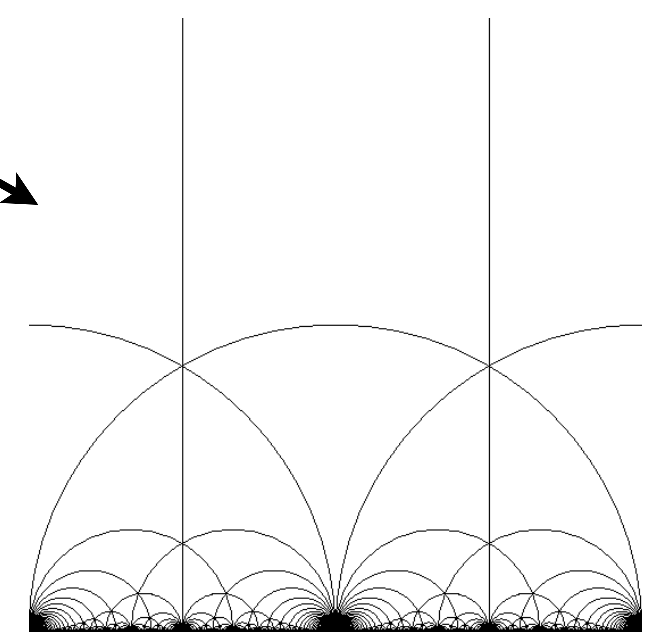
Calabi-Yau Avatars of Mathieu Moonshine





Shamit Kachru, Strings 2013, Seoul

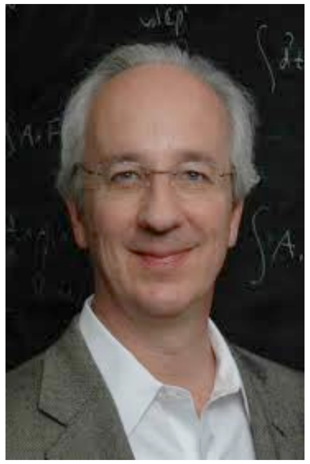
Based (in part) on a very recent paper with this cast of characters:

M. Cheng

X. Dong

J. Duncan

J. Harvey



T. Wrase

This talk will be organized in three short sections:

I. What is moonshine?

II. Enter K3 and BPS States

III. String duality, Calabi-Yau threefolds, and modular forms

The first two are just review of material that may be unfamiliar to many of you.

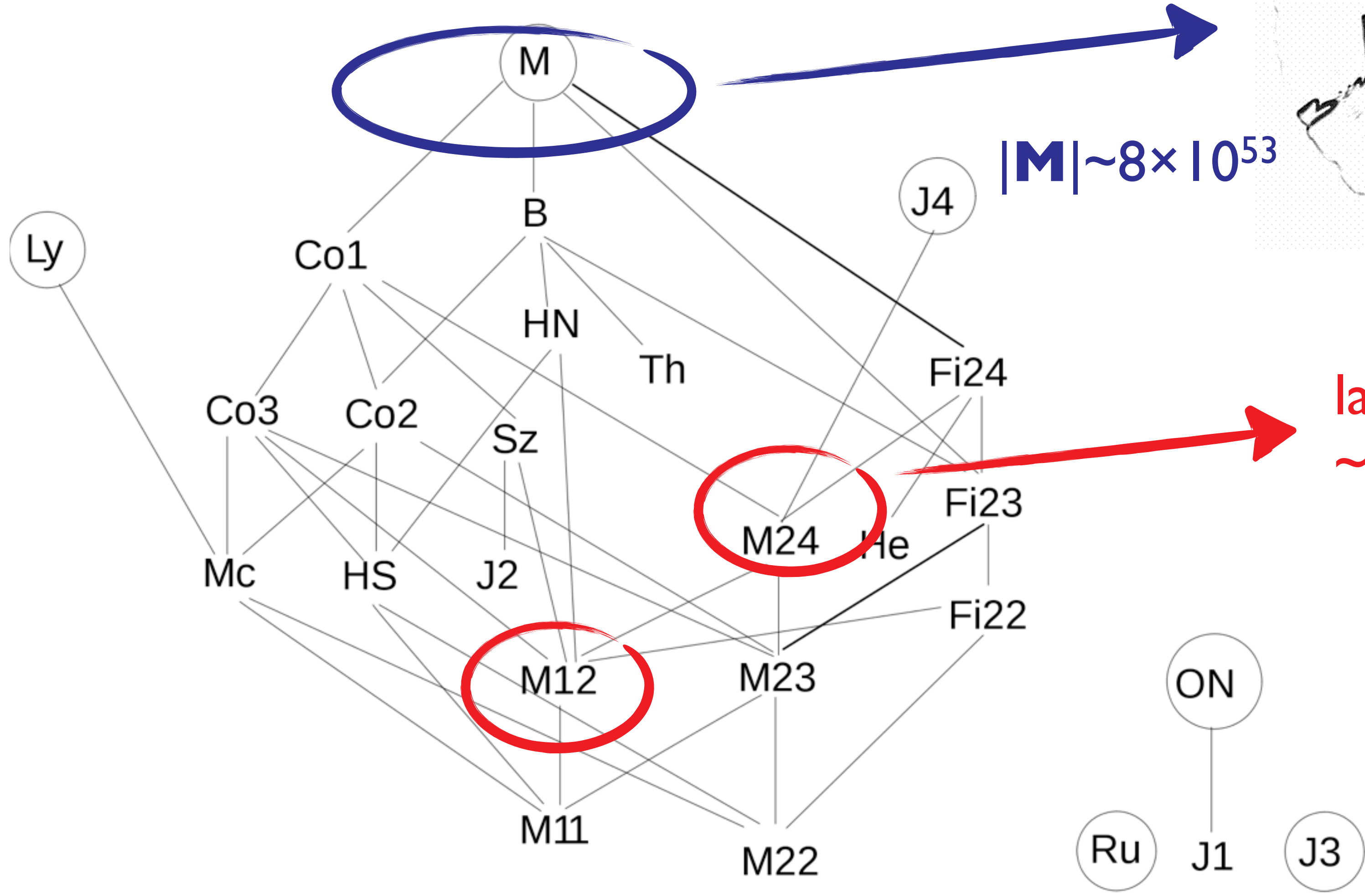
1. Monstrous Moonshine

Moonshine is a still mysterious relationship between natural objects from two distinct realms of mathematics: finite groups and modular forms.

By ~1980, the classification of simple finite groups was in sight. In the final result, there are 18 infinite series and 26 sporadics. The last and largest of the latter, the Fischer-Griess Monster, was just discovered and being explored in the early 1980s.

Sporadic Groups

The 26 finite simple groups that don't come in ∞-families.

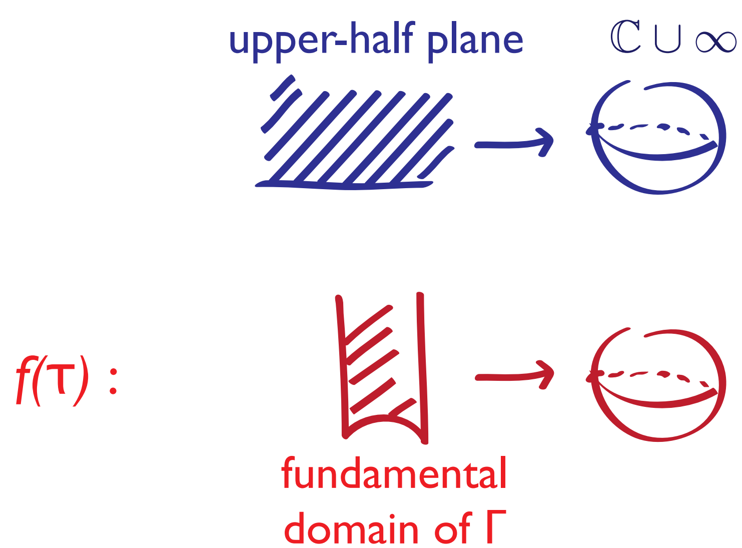


$|\mathbf{M}| \sim 8\times 10^{53}$

largest Mathieu group
$\sim 2\times 10^{9}$

The first nontrivial representation of the Monster occurs in dimension 196,883.

John McKay noticed an interesting coincidence. Recall the definition of modular functions and forms:



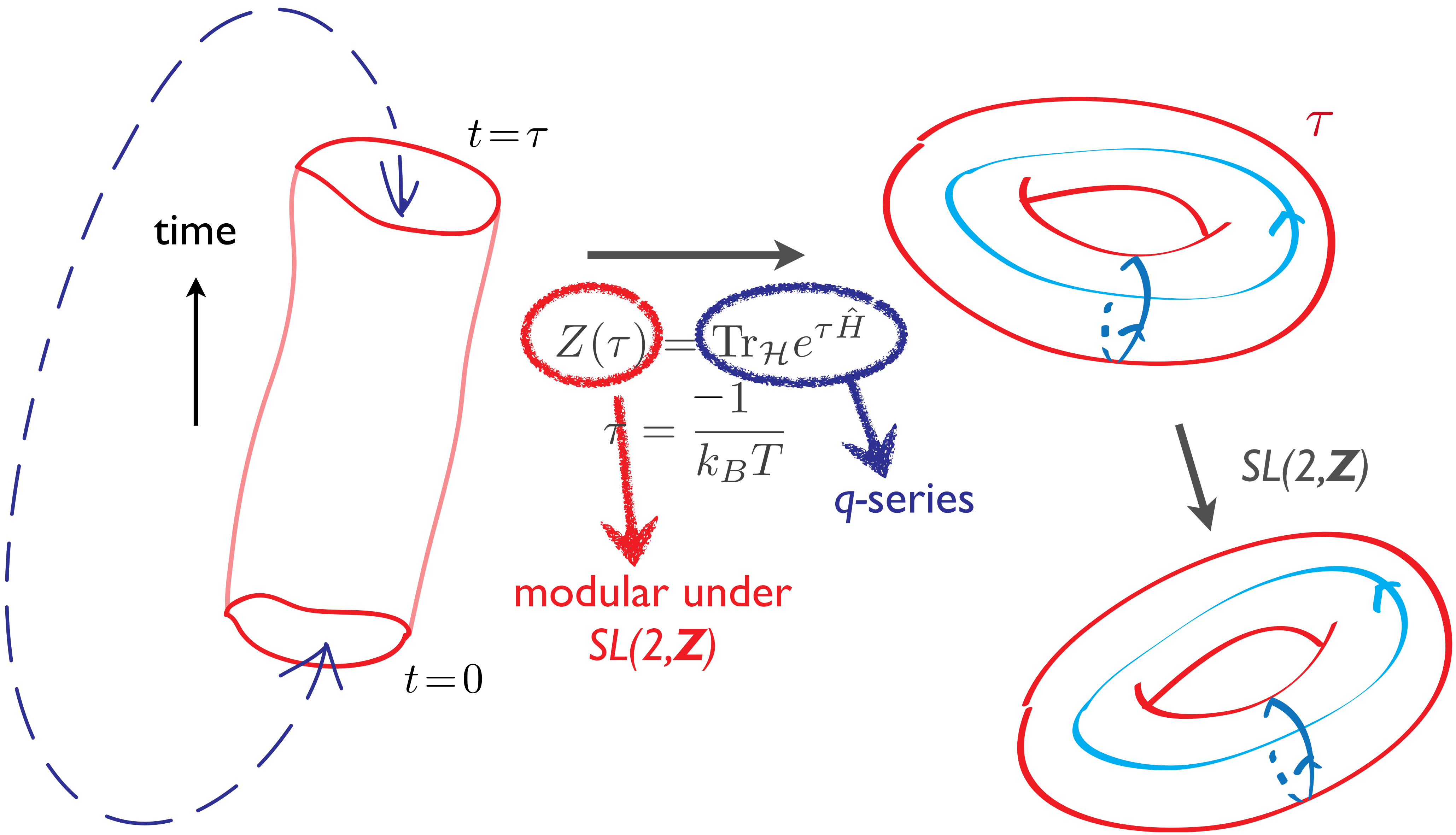
$$\tau \to \frac{a\tau+b}{c\tau+d} \equiv A\cdot\tau$$

modular function: $f(A\cdot\tau) = f(\tau)$

modular form: $f(A\cdot\tau) = (c\tau+d)^k f(\tau)$

These beasts pop up everywhere in string theory, for two reasons: worldsheet modular invariance and space-time S-duality symmetries.

eg. The *SL(2,**Z**)* symmetry of the partition function.



A basic result says that any modular function can be written as a rational function of

$$j(\tau) = \frac{1}{q} + 744 + 196,884\ q + 21,493,760\ q^2 + \cdots$$

$$q = e^{2\pi i \tau}$$

What McKay noticed was:

$$196,884 = 196,883 + 1$$

$$21,493,760 = 21,296,876 + 196,883 + 1$$

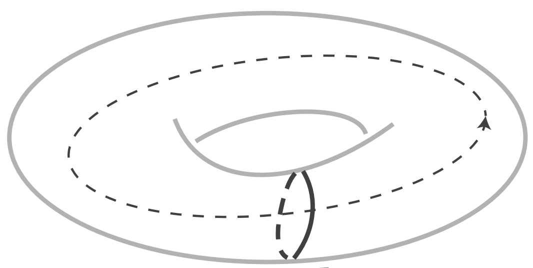
dims of irreps of Monster!

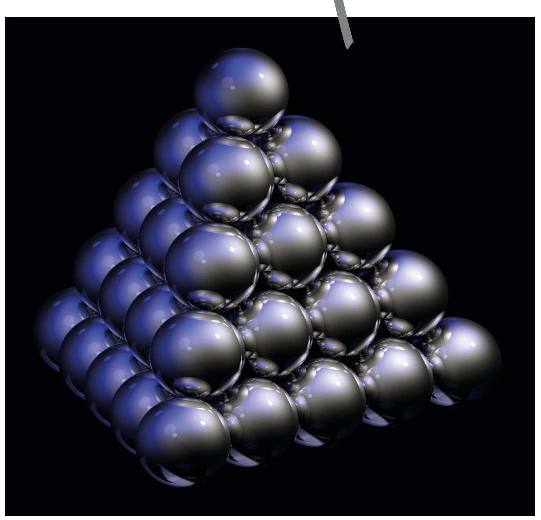
$$\cdots$$

At some level, one can answer the question of why as follows:

Frenkel, Lepowsky, Meurman;
Dixon, Ginsparg, Harvey;
Borcherds

Strings in the Leech lattice background:

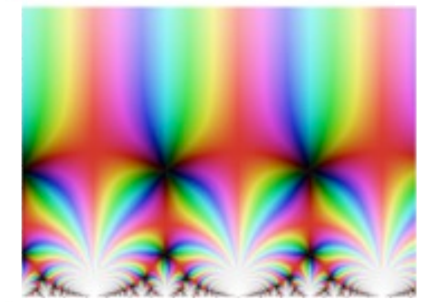




R^{24}/Λ_{Leech}

Modular Symmetry

The Partition Function = $J(\tau)$



Monstrous Moonshine

Sporadic Symmetry

Λ_{Leech}

$Co_1 \xrightarrow{Z/2}$ ***Monster***

Given this field theory, if we compute the partition function twisted by elements of the symmetry group:

$$h\ \underset{g}{\square} \quad \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \quad h^d g^c\ \underset{g^a h^b}{\square}$$

we should still get a modular form for a subgroup of the modular group that preserves the form of the BC.

Twisting by g, we'd expect to find, instead of j, the McKay-Thompson series:

$$ch_\rho(g) = Tr(\rho(g)), \quad g \in M$$

$$T_g(\tau) = ch_{V_{-1}}(g)\ q^{-1} + \sum_{i=1}^{\infty} ch_{V_i}(g)\ q^i$$

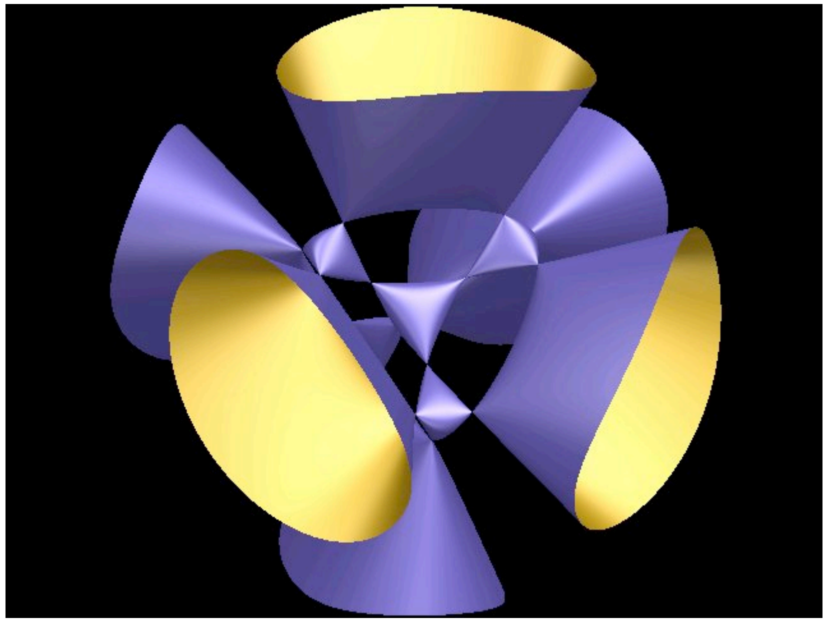
For each conjugacy class in M (there are 194), we get such a series.

The fact that these are modular with the expected properties is strong evidence for the conjectured decomposition of the Hilbert space in terms of representations of the Monster, and would be testable even without the explicit construction of the CFT.

Lesson: When a modular form with suggestive coefficients appears, should check twining genera also to really convince oneself any kind of moonshine is at work...

While the story relating Monstrous Moonshine to a particular 2d CFT is beautiful to be sure, it has seemed a peculiarity.

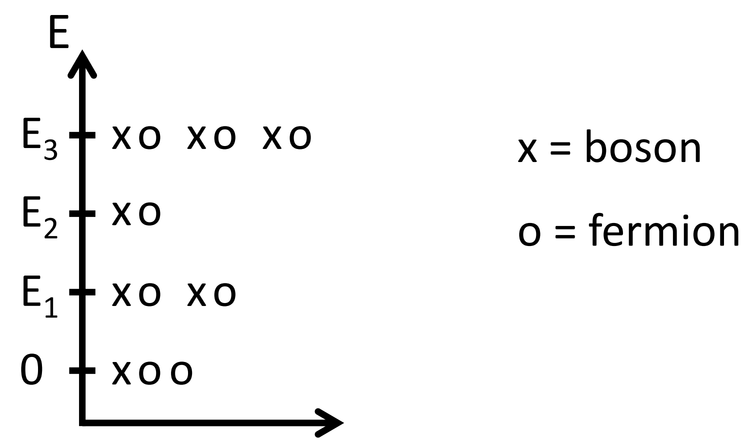
II. Enter K3 and BPS States



K3 serves as the simplest nontrivial example of Calabi-Yau compactification. It has also played a central role in string dualities since the mid 1990s.

The K3 sigma model is in general a highly nontrivial interacting (4,4) supersymmetric theory. Index information can be useful when the theory cannot be solved.

The most famous example:



As you vary couplings, the finite energy states can move around at will. But notice that states can join or leave E=0 only in pairs! So:

$$\mathrm{Tr}(-1)^F = n_B - n_F$$

is an invariant, the "Witten index."

The hero of this part of our story is a more refined invariant that can be defined in 2d theories with at least (0,2) supersymmetry: the elliptic genus.

Lets assume the left movers and right movers both enjoy N=2 SUSY. The N=2 algebra requires:

$$\{Q_R^+, Q_R^-\} = L_0 = H_R$$

The one-loop partition function is:

$$Z(q, \gamma_L, \gamma_R) = Tr_{\mathcal{H}}(-1)^F q^{H_L} \bar{q}^{H_R} e^{i\gamma_L J_L + i\gamma_R J_R}$$

Now, the N=2 algebra has irreducible highest weight representations (think of Verma modules...). Let us say that these are denoted by

$$R_\alpha, \bar{R}_\alpha$$

for the left/right N=2. Then we can write:

$$Z = \sum_\alpha Tr_{R_\alpha}(-1)^{F_L}\; q^{H_L}\; e^{i\gamma_L J_L}\; Tr_{\bar{R}_\alpha}(-1)^{F_R}\bar{q}^{H_R}e^{i\gamma_R J_R}$$

However, except in very special CFTs, we cannot solve for the full partition function. (It contains e.g. the spectrum of the space-time theory).

So we can consider sacrificing some information to get an index, which will be computable.

* If we set

$$\gamma_L = \gamma_R = 0$$

we basically get the Witten index. Not a lot of info.

* However, we can consider setting only say

$$\gamma_R = 0$$

Then the right-moving bit of the sum becomes:

$$Tr_{\bar{R}_\alpha} (-1)^{F_R} \, \bar{q}^{H_R}$$

So we only get contributions from right-moving ground states! (The elliptic genus can be defined for theories with only right-moving N=2 SUSY, since its all we use here).

It reduces then to the function:

$$Z(q, \gamma_L, 0) = \sum_\alpha' \mathrm{Tr}_{R_\alpha} \ (-1)^{F_L} \ q^{H_L} \ e^{i\gamma_L J_L} \ (-1)^{F_R}$$

This is clearly a more refined invariant than the Witten index - in fact, it is a modular form.

In the K3 sigma model, the elliptic genus counts (a weighted sum of) space-time BPS states arising in the string compactification to 6d.

The elliptic genus was computed by Eguchi, Ooguri, Taormina and Yang in 1989. It is:

$$\phi_{K3}(\tau, z) = \mathrm{Tr}_{RR}\left(q^{L_0 - \frac{c}{24}}\ y^{J_L}\ (-1)^F\right)$$

$$q = e^{2\pi i \tau},\ y = e^{2\pi i z}$$

The result they found:

$$\phi_{K3}(\tau, z) = 8\ \sum_{i=2}^{4}\ \frac{\theta_i(\tau, z)^2}{\theta_i(\tau, 0)^2}$$

Representations of the N=4 worldsheet algebra are labelled by conformal spin h and an SU(2) quantum number l. Expanding the elliptic genus in terms of multiplets of the left-moving N=4 yields:

$$\phi_{K3}(\tau,z) = 20\ ch^{\text{short}}_{1/4,0}(\tau,z) - 2\ ch^{\text{short}}_{1/4,1/2}(\tau,z) + \sum_{n=1}^{\infty} A_n\ ch^{\text{long}}_{1/4+n,1/2}(\tau,z)$$

The values of the As are given by:

$$A_1 = 90 = \mathbf{45} + \overline{\mathbf{45}}$$

$$A_2 = 462 = \mathbf{231} + \overline{\mathbf{231}}$$

$$A_3 = 1540 = \mathbf{770} + \overline{\mathbf{770}}$$

dims of irreps of M24!

Eguchi, Ooguri, and Tachikawa noticed this in 2010, and conjectured a "Mathieu Moonshine" relating K3 compactification to the sporadic group M24.

Much further work has elucidated and generalized the EOT conjecture.

Cheng; Gaberdiel, Hohenegger, Volpato; Eguchi, Hikami; Cheng, Duncan, Harvey; Taormina, Wendland; Gannon; ...

It needs to be emphasized that as yet, there is no explicit realization of a physically relevant CFT which "explains" the full moonshine. In fact, there is a proof that (4,4) sigma models on K3 never admit the full M24 symmetry.

Here, we ask a different question. Does some evidence of moonshine persist in vacua with less supersymmetry?

We focus on 4d N=2 string vacua.

III. String duality, Calabi-Yau threefolds, and modular forms

Such vacua arise in two simple avatars, related by duality:

SK, Vafa;
Ferrara, Harvey, Strominger, Vafa

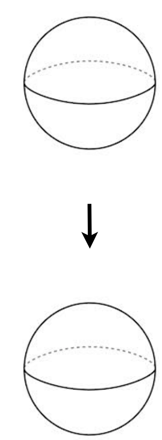
Heterotic strings on K3xT2

$$\int_{K3} c_2(V_1) + c_2(V_2)$$

$$\equiv n_1 + n_2 = 24$$

dilaton S

Type IIA on Calabi-Yau threefolds



elliptic fibration over

$F_n, \; n_1 = 12 + n, n_2 = 12 - n$

size of base P^1

They can be lifted to 6d dualities in F-theory.

Morrison, Vafa

There is a web of such vacua with various possible unHiggsed gauge groups and Coulomb branches, familiar also from studies of N=2 field theory. Dual descriptions of the heterotic gauge groups involve singular Calabi-Yau spaces.

Diagram 1: Higgs Tree

$$
\begin{array}{ccccccccc}
SU(3)_2 & \leftarrow & Sp(3) & \leftarrow & SU(6) & \leftarrow & SO(12) & & \\
 & & & & & & \downarrow & & \\
 & & & & \downarrow & & SO(11) & \leftarrow & E_7 \\
 & & & & & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & SU(5) & \leftarrow & SO(10) & \leftarrow & E_6 \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & SO(9) & \leftarrow & F_4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & SO(8) & & \\
 & & & & & & \downarrow & & \\
\downarrow & & Sp(2) & \leftarrow & SU(4) & \leftarrow & SO(7) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & \downarrow & & SU(3) & \leftarrow & G_2 & & \\
 & & & & \downarrow & & & & \\
SU(2)_2 & \leftarrow & SO(4) & \rightarrow & SU(2) & & & &
\end{array}
$$

Bershadsky, Intriligator, SK, Morrison, Sadov, Vafa

A good quantity to look at is the "new supersymmetric index" of the heterotic (0,4) conformal field theory:

$$\mathcal{Z}_{new} = \frac{1}{\eta(q)^2} \mathrm{Tr}_R J_0 e^{i\pi J_0} q^{L_0 - c/24} \bar{q}^{\overline{L_0} - \bar{c}/24}$$

Cecotti, Fendley, Intriligator, Vafa

This quantity is particularly interesting because one can show that "morally":

$$\mathcal{Z}_{new} = -2i \left[\sum_{\text{BPS vectors}} q^{\Delta} \bar{q}^{\bar{\Delta}} - \sum_{\text{BPS hypers}} q^{\Delta} \bar{q}^{\bar{\Delta}} \right] .$$

Harvey, Moore

Beyond morals, it also shows up in the 1-loop threshold corrections to the space-time theory:

Antoniadis, Gava, Narain

$$\Delta_{\text{gauge/grav}} = \int \frac{d^2\tau}{\tau_2}\left[-\frac{i}{\eta(q)^2}\mathrm{Tr}_R\left(J_0 e^{i\pi J_0} q^{L_0-c/24}\bar{q}^{\overline{L_0}-\bar{c}/24}F_{\text{gauge/grav}}\right) - b_{\text{gauge/grav}}\right]$$

$$F_{\text{gauge}} = Q^2 - \frac{1}{8\pi\tau_2}, \qquad F_{\text{grav}} = E_2(q) - \frac{3}{\pi\tau_2} \equiv \hat{E}_2(q)$$

$$E_2(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$$

It follows that we expect

$$-\tau_2\frac{i}{\eta(q)^2}\mathrm{Tr}_R\left(J_0 e^{i\pi J_0} q^{L_0-c/24}\bar{q}^{\overline{L_0}-\bar{c}/24}F_{\text{gauge/grav}}\right)$$

to be modular invariant.

Focusing on the case of the gravitational threshold:

* Turn off all Wilson lines

* Then the index further factorizes as:

$$\mathcal{Z}_{new} = -2i \frac{1}{\eta(q)^2} \frac{\Theta_{\Gamma_{2,2}}}{\eta(q)^2} \mathcal{G}_{K3}$$

$$\begin{aligned} \Theta_{\Gamma_{2,2}}(q, \bar{q}; T, U, \bar{T}, \bar{U}) &= \sum_{p \in \Gamma_{2,2}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} = \sum_{p \in \Gamma_{2,2}} q^{\frac{1}{2}(p_L^2 - p_R^2)} e^{-2\pi\tau_2 p_R^2} \\ &= \sum_{m_i, n_i \in \mathbb{Z}} e^{2\pi i \tau (m_1 n_1 + m_2 n_2) - \frac{\pi\tau_2}{T_2 U_2} |TUn_2 + Tn_1 - Um_1 + m_2|^2} \end{aligned}$$

It follows that $\frac{\mathcal{G}_{K3}}{\eta(q)^4}$ has weight -2.

Using the facts that :

See also:
Lopes Cardoso, Curio, Lust;
Stieberger;....

* $\mathcal{Z}_{new}$ has a 1/q pole

* This must come from $\frac{\mathcal{G}_{K3}}{\eta^4}$ because the torus sum has only positive powers

* Then $\eta^{20}\mathcal{G}_{K3}$ must be a holomorphic modular form of weight 10

we find that

$$\mathcal{G}_{K3} = \frac{E_4(q)E_6(q)}{\eta^{20}(q)}$$

independent of the choice of gauge bundles on K3.

The normalization can be fixed by e.g. requiring cancellation of the 6d gravitational anomalies.

This universal structure contains a factor:

$$G(q) = -4E_6/\eta^{12}$$

$$G(q) = 24 g_{h=1/4,l=0}(q) + g_{h=1/4,l=1/2}(q) \sum_{n=0}^{\infty} A_n q^n$$

$$\begin{aligned} g_{h=1/4,l}(q) &= \left(\mathrm{ch}_s^{SO(12)} + \mathrm{ch}_c^{SO(12)}\right) \mathrm{ch}_{h=1/4,l}(q,-1) \\ &+ q^{1/4}\left(\mathrm{ch}_b^{SO(12)} + \mathrm{ch}_v^{SO(12)}\right) \mathrm{ch}_{h=1/4,l}(q,-q^{\frac{1}{2}}) \\ &- q^{1/4}\left(\mathrm{ch}_b^{SO(12)} - \mathrm{ch}_v^{SO(12)}\right) \mathrm{ch}_{h=1/4,l}(q,q^{\frac{1}{2}}) \,. \end{aligned}$$

Importantly, as with the K3 elliptic genus,

$$A_n = -2, 90, 462, 1540, 4554, 11592, \ldots \,.$$

dims of irreps of M24!

How does this modular form manifest itself in the dual Calabi-Yau geometries?

* The 1-loop threshold corrections determine the 1-loop prepotential in the heterotic theory.

* In the dual Calabi-Yau compactification, one should then focus on the prepotential in the limit

$$\text{Perturbative heterotic limit}: t_{B_1} \to \infty,\ \ q_{B_1} \equiv e^{-2\pi t_{B_1}} \to 0\ .$$

The universal heterotic result showing the nice degeneracies should be (and is!) visible in curve counts in type IIA string theory.

We make use of recent results in the mathematical physics literature. Expanding:

Alim, Scheidegger;
Klemm, Manschot, Woschke

$$F^{(g)}(q^A) = \sum f^{(g)}_{k,l}(q_F)\, q^k_{B_1} q^l_{B_2}$$

it has been shown that the coefficient "functions" are of the form

$$f^{(g)}_{k,l}(q_F) = \left(\frac{q_F^{\frac{1}{24}}}{\eta(q_F)}\right)^{2p(k,l)} P_{2g-2+p(k,l)}(E_2(q_F), E_4(q_F), E_6(q_F)),$$

$P_{2g-2+p(k,l)}$ is a quasi-modular form of weight $2g-2+p(k,l)$

$$p(k,l) = \frac{k}{2}\int_M c_2(M)\wedge J_2 + \frac{l}{2}\int_M c_2(M)\wedge J_1 \ .$$

They furthermore satisfy interesting recursion relations. E.g. for n=0,1,2:

$$
\begin{aligned}
\frac{\partial f^{(g)}_{k,l}}{\partial E_2} &= \frac{1}{24}\sum_{h=0}^{g}\sum_{s=0}^{k}\sum_{t=0}^{l}(ns(k-s)-s(l-t)-t(k-s))f^{(g-h)}_{s,t}f^{(h)}_{k-s,l-t} \\
&\quad -\frac{1}{24}(2kl+(n-2)k-2l-nk^2)f^{(g-1)}_{k,l}\,,
\end{aligned}
$$

where the terms with k=0 can be compared to the perturbative heterotic string.

The simplest non-trivial term of interest satisfies:

$$
\frac{\partial f^{(0)}_{0,1}}{\partial E_2} = 0\ .
$$

Using this together with the fact that

$$\int c_2(M) \wedge J_1 = 24$$

for these K3-fibered Calabi-Yau threefolds, we find:

$$f_{0,1}^{(0)}(q_F) = -\frac{1}{4\pi^3} \frac{q_F E_4(q_F) E_6(q_F)}{\eta(q_F)^{24}}$$

The same modular form appears!

It previously appeared in an integrand of a 1-loop amplitude. Here, it appears as a function of space-time moduli in the type II model.

However, computing the one-loop heterotic prepotential in the same limit, using the result for the new supersymmetric index, one indeed finds: c.f. Kaplunovsky, Louis; de Wit, Kaplunovsky, Louis, Lust; Harvey, Moore

$$F^{het} = \ldots - \frac{1}{4\pi^3} q_F q_{B_2} \sum_{l \geq -1} c(l) q_F^l + \mathcal{O}(q_{B_1}, q_{B_2}^2) \ .$$

Here:

$$\frac{E_4 E_6}{\eta^{24}} = \sum_{m \geq -1} c(m) q^m = \frac{1}{q} - 240 + \cdots$$

Perfect agreement!

Summary of other results:

* One can check twining genera for certain simple M24 conjugacy classes in orbifold limits of models with various n. We have some positive results.

* One can find evidence for moonshine also in heterotic models with all instantons in one E8 and generic Wilson lines in the other.

* Results indicating moonshine in higher genus curve counts can also be seen.

Very many questions remain. For more details see the paper that appeared today!